Universidad Politécnica
de Madrid

Universidad Nacional de
San Luis

Approximation algorithms for Art Gallery problems

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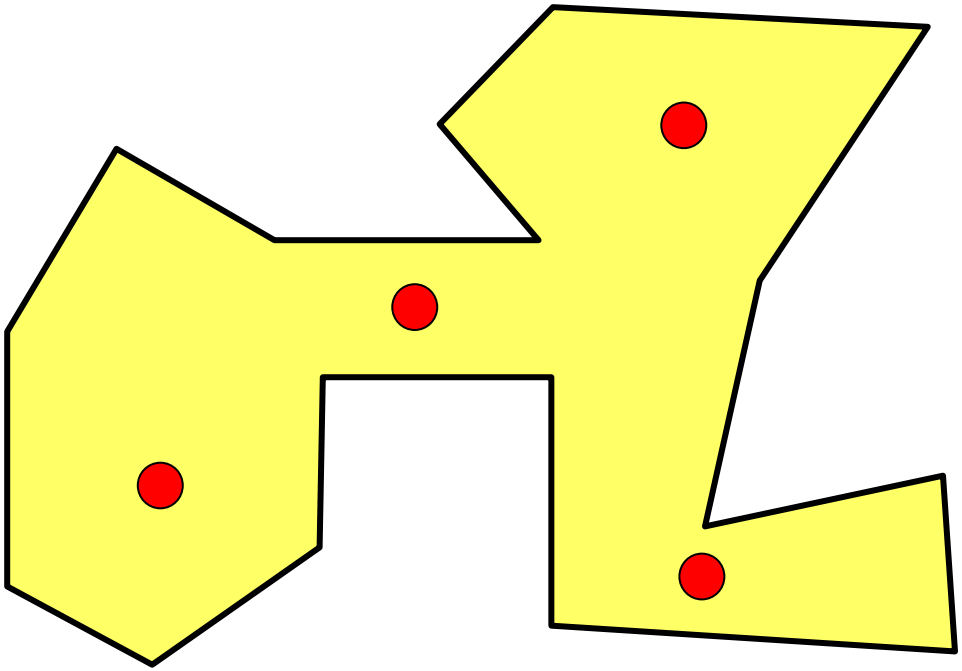
How many guards?

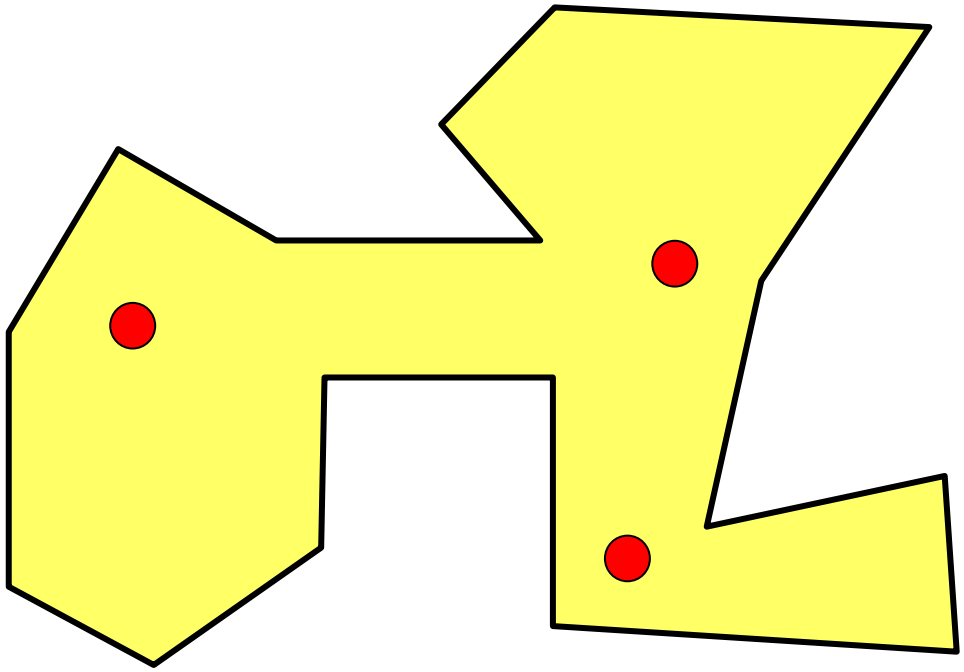


Victor Klee, 1973

How many lights? How many guards?

ART GALLERY Problem

ART GALLERY Problem



Minimize the number of guards for P g(P)

ALGORITHMIC PROBLEM

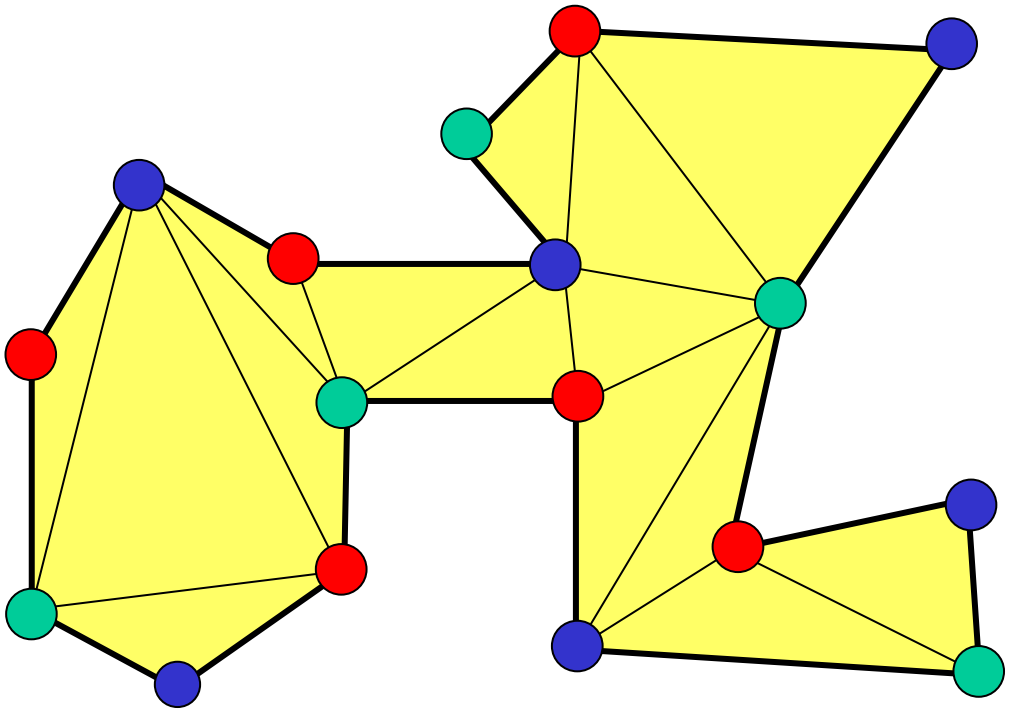
The problem is NP-hard Aggarwal, 84
Lee-Lin, 86

Find g(n), the minimum number of guards that are always sufficient and sometimes necessary for guarding any polygon with n vertices

$g(n) = \lfloor n/3 \rfloor$

COMBINATORIAL PROBLEM

ART GALLERY Problem



COMBINATORIAL PROBLEM

Chvátal, 1975

Fisk, 1978

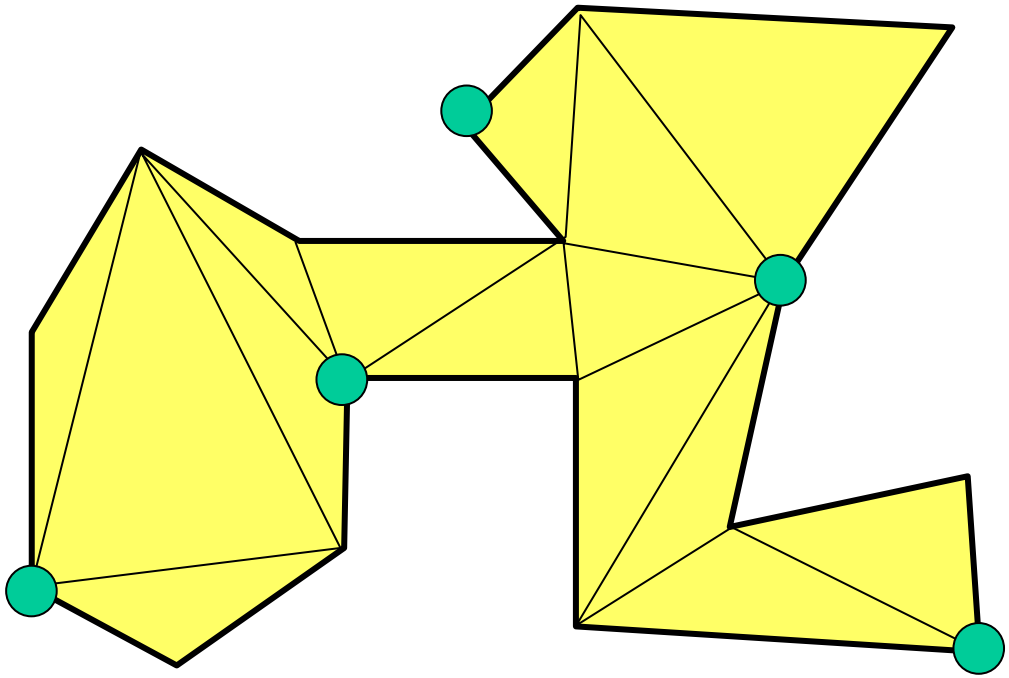
R + A + V = n

1) Triangulate the polygon

2) Coloring the vertices with three colours

3) Place one guard at each vertex of the smallest chromatic class

ART GALLERY Problem



COMBINATORIAL PROBLEM

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$R + A + V = n$

$$V \leq \left\lfloor \frac{n}{3} \right\rfloor$$

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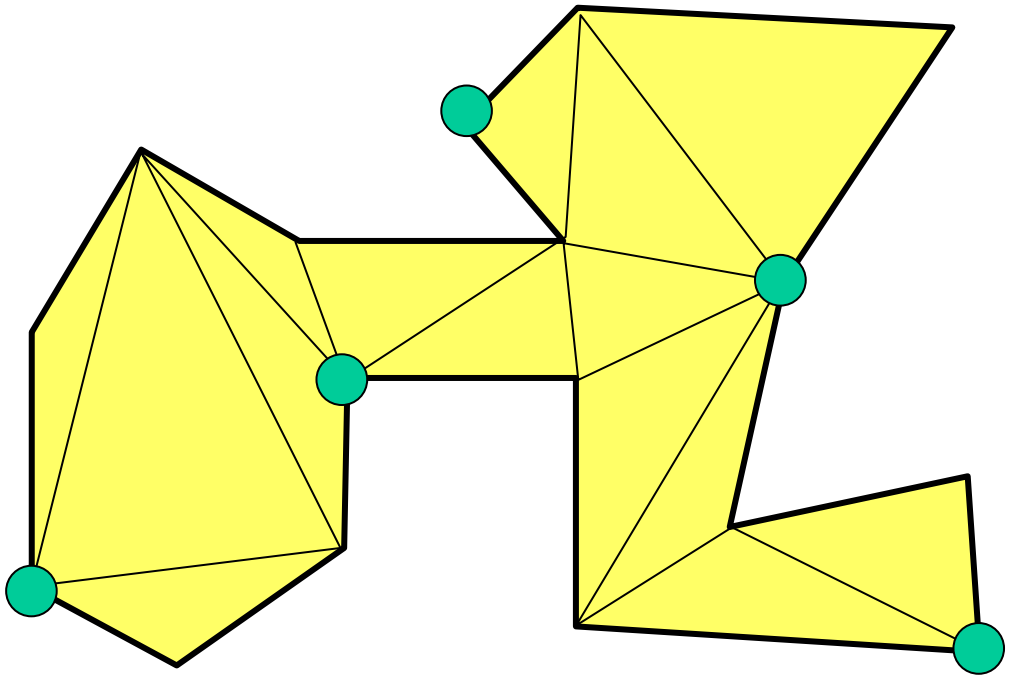
ART GALLERY Problem



COMBINATORIAL PROBLEM

Chvátal, 1975

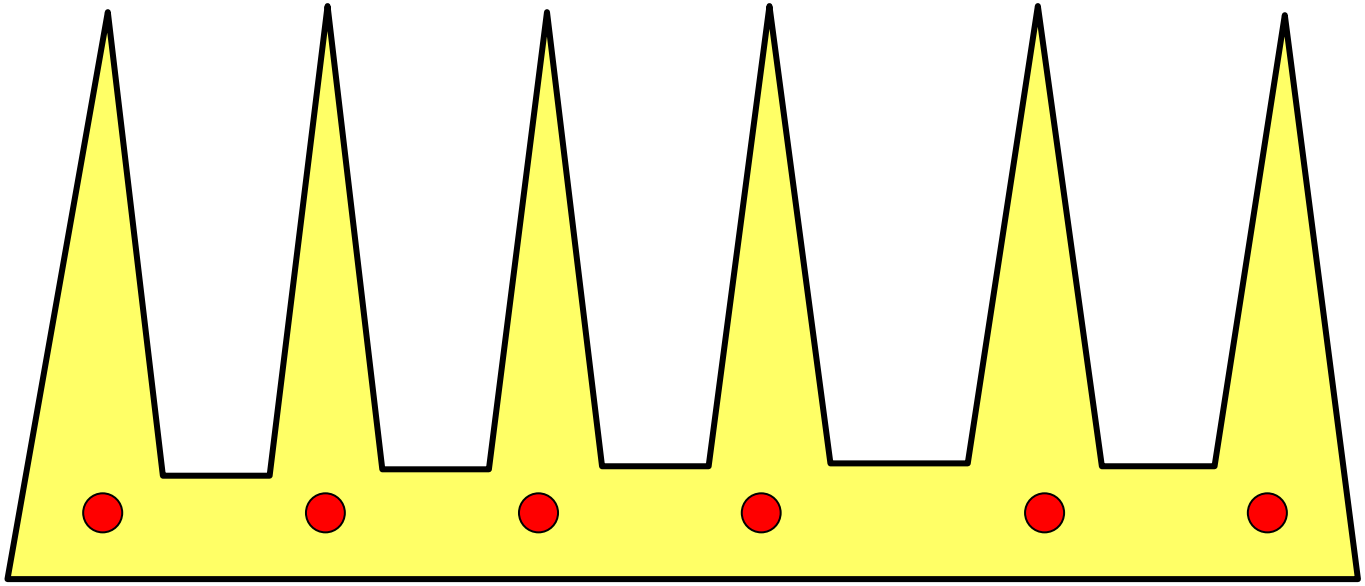
Fisk, 1978

$R + A + V = n$

$V \leq \left\lfloor \frac{n}{3} \right\rfloor$

"Any polygon with n vertices can be guarded with $\left\lfloor \frac{n}{3} \right\rfloor$ guards"

ART GALLERY Problem



Theorem

$\lfloor n/3 \rfloor$ guard are always sufficient and occasionally necesary to illuminate a polygonal art gallery with n vertices

ART GALLERY Problem

VARIANTS

→ Which are the objects to guard?

- orthogonal polygons
- polygons with holes
- interior, walls, exterior …

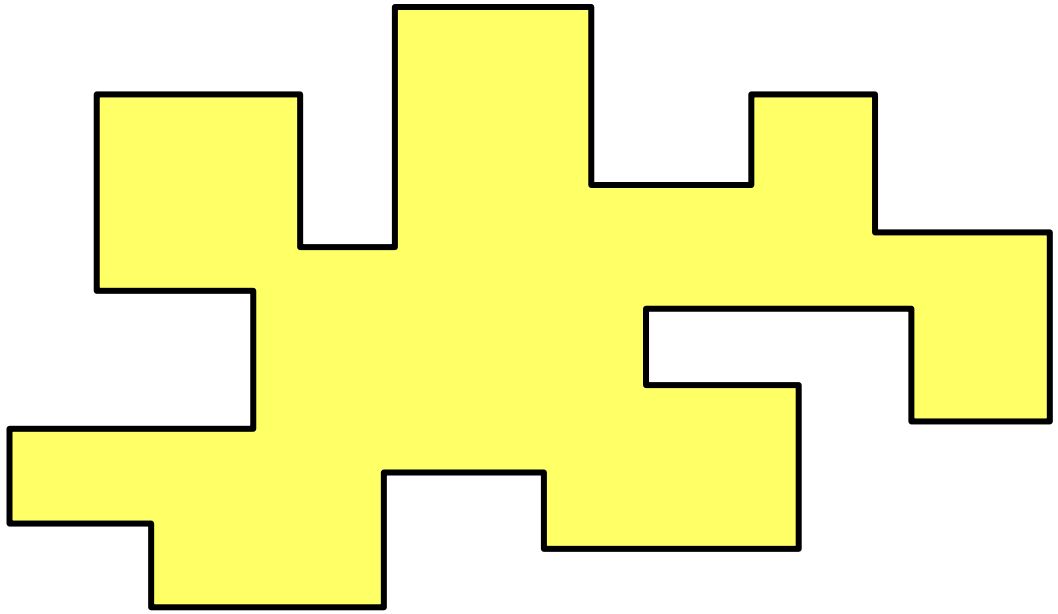
→ How are the objects illuminated?

- from vertex (vertex-guards), interior points (guards)
- mobile guards (edge guards, diagonal guards)
- floodlights
- through the walls …

ART GALLERY Problem

Minimizing the number of guards for a polygon P is NP-hard

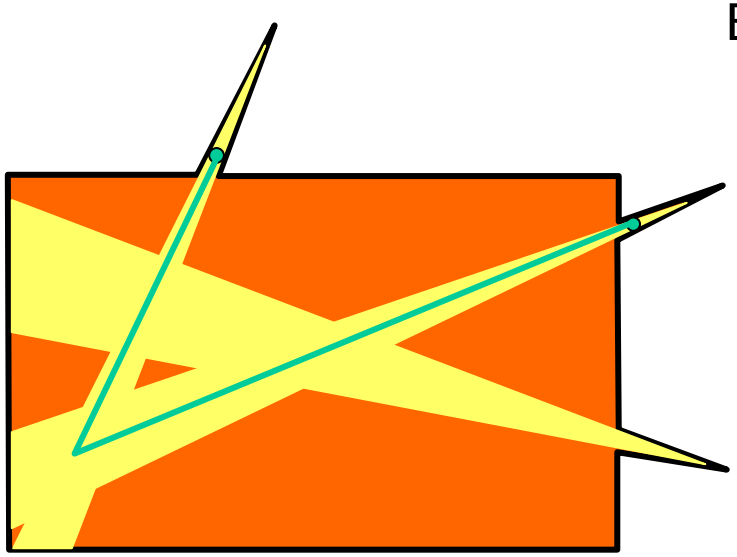
- Simple polygons — vertex guards, (interior) guards
  Lee-Lin, Aggarwal, 84
- Orthogonal polygons — vertex guards, (interior) guards
  Schuchardt, Hecker, 95



ART GALLERY Problem

Minimizing the number of guards for a polygon P is NP-hard

- Simple polygons — vertex guards, (interior) guards
  Lee-Lin, Aggarwal, 84
- Orthogonal polygons — vertex guards, (interior) guards
  Schuchardt, Hecker, 95
- Polygons with link-diameter 2
  Broden, Hammar, Nilsson, 01



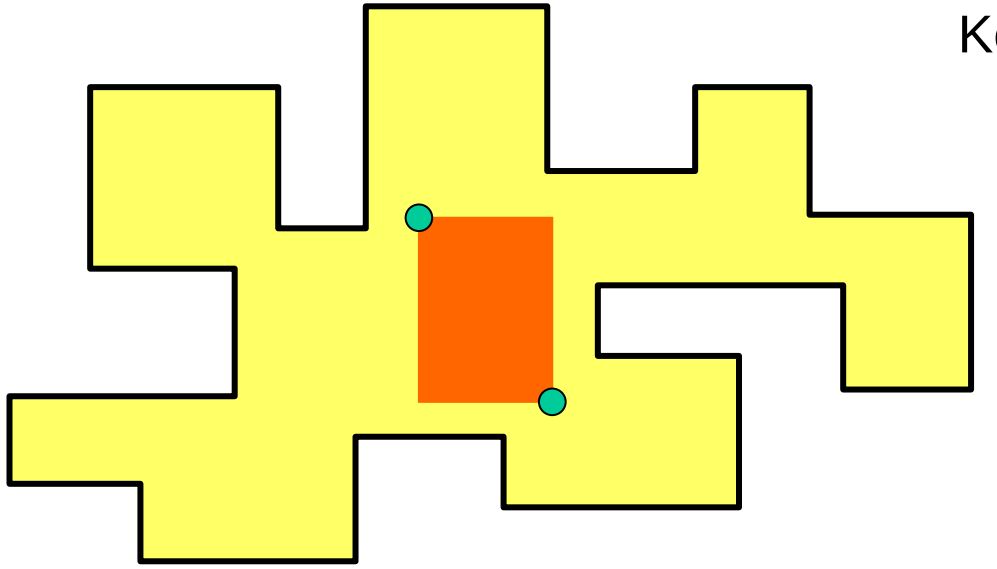
ART GALLERY Problem

Minimizing the number of guards for a polygon P

There is an exact algorithm for **ONE** type of guarding

Orthogonal polygon

Rectangle visibility



Keil, Worman, 06

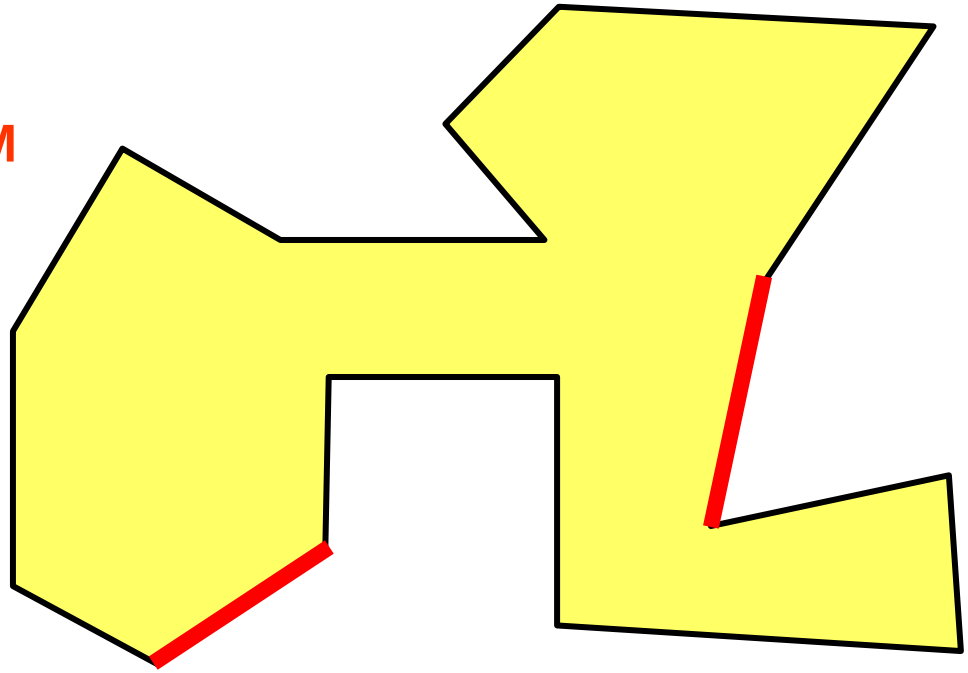
$O(n^{17})$

EDGE GUARDING

How many edge-guards?

COMBINATORIAL PROBLEM

Conjecture (Toussaint, 1981)



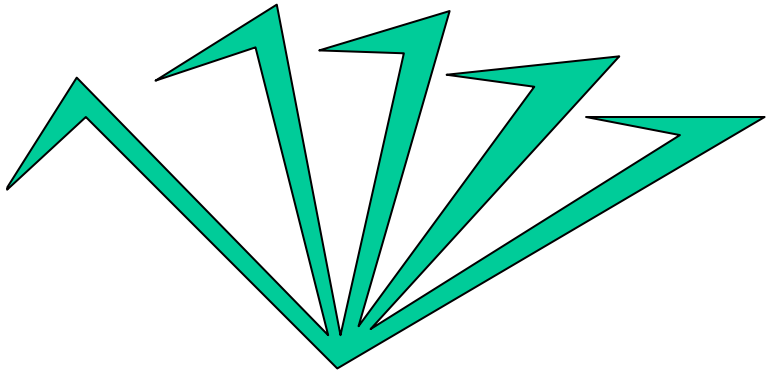
⌊n/4⌋ edge-guards are always sufficient and sometimes necessary for guarding any polygon with n vertices

EDGE GUARDING

How many edge-guards? Conjecture (Toussaint, 1981)

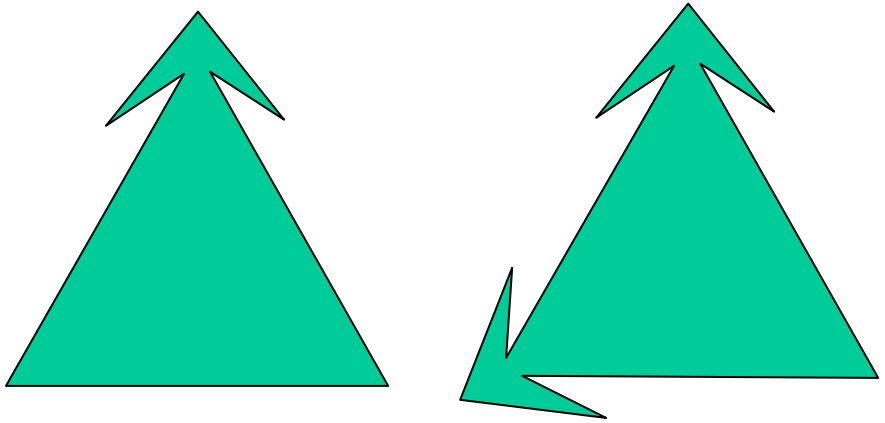
COMBINATORIAL PROBLEM

$\lfloor n/4 \rfloor$ edge-guards are always sufficient and sometimes necessary for watching any polygon with n edges



necessary

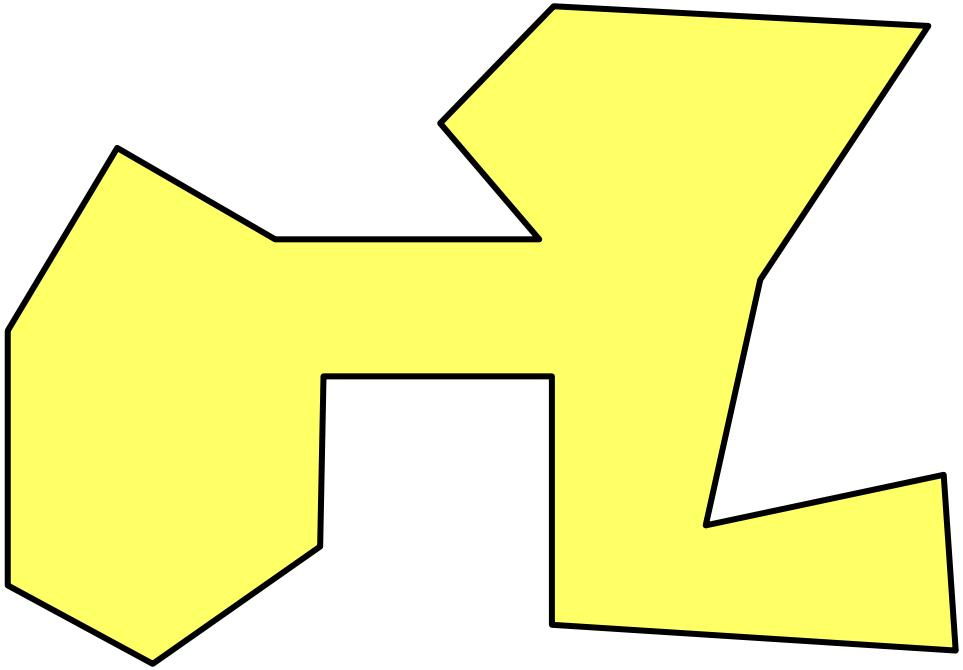
... except these polygons



EDGE GUARDING

ALGORITHMIC PROBLEM

Minimize the number of edge-guards for P

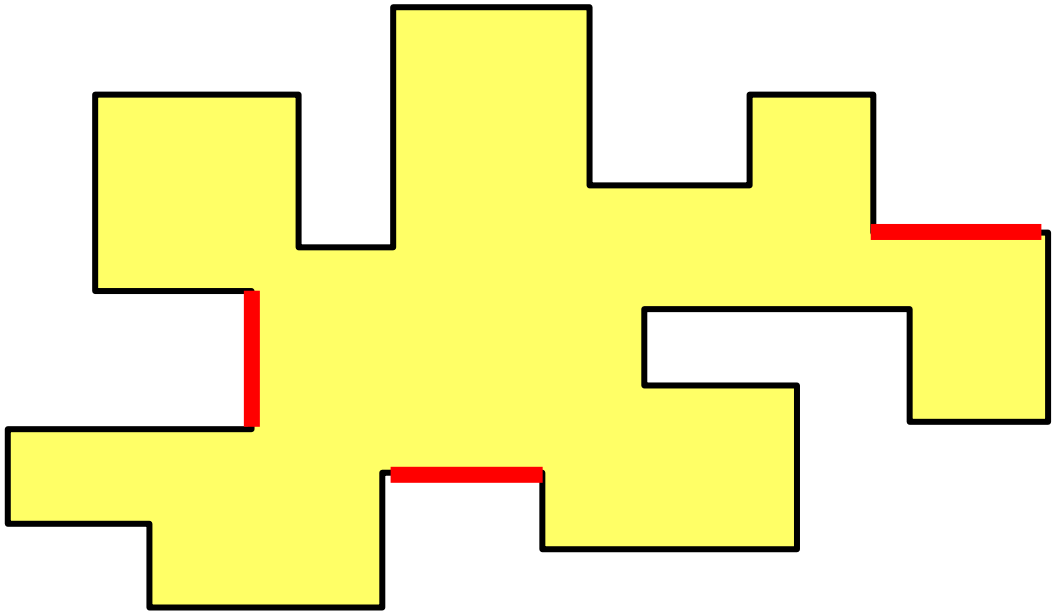


NP-hard
Lee, Lin, 84

EDGE GUARDING

Orthogonal polygons

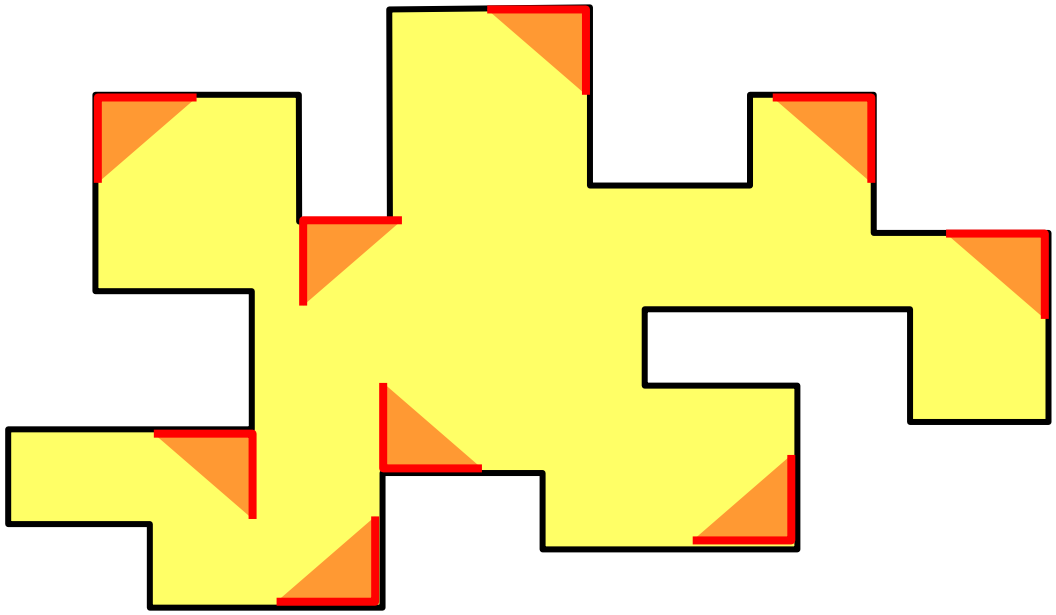
Minimize the number of edge-guards for P



NP-hard

Orthogonal polygons

ORTHOGONAL FLOODLIGHT



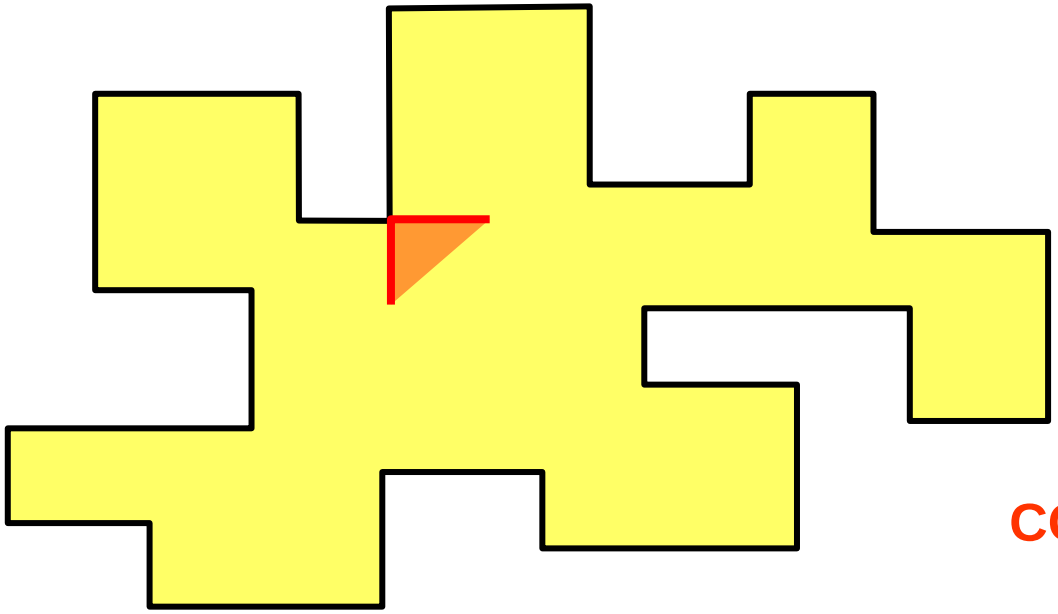
ALGORITHMIC PROBLEM

Minimizing the number of orthogonal vertex floodlights for a polygon P is NP-hard?

OPEN PROBLEM

Orthogonal polygons

ORTHOGONAL FLOODLIGHT

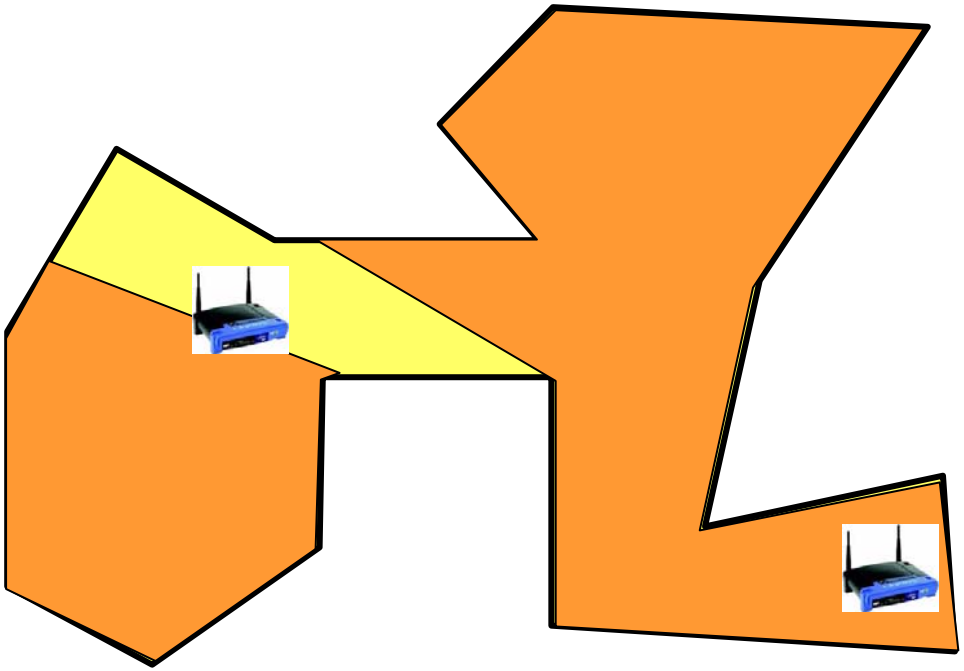


COMBINATORIAL PROBLEM

Any orthogonal polygon can always be illuminated with at most $\left\lfloor \frac{3n-4}{8} \right\rfloor$ orthogonal vertex floodlights. The bound is tight.
Estivill, Urrutia, 94

Urrutia, 06

VISIBILITY THROUGH THE WALLS



Wireless router!! 

If the signal can cross two walls, we need two routers in this polygon

Minimizing the number of k-routers for covering a polygon P is NP-hard?

k-router can cross k walls

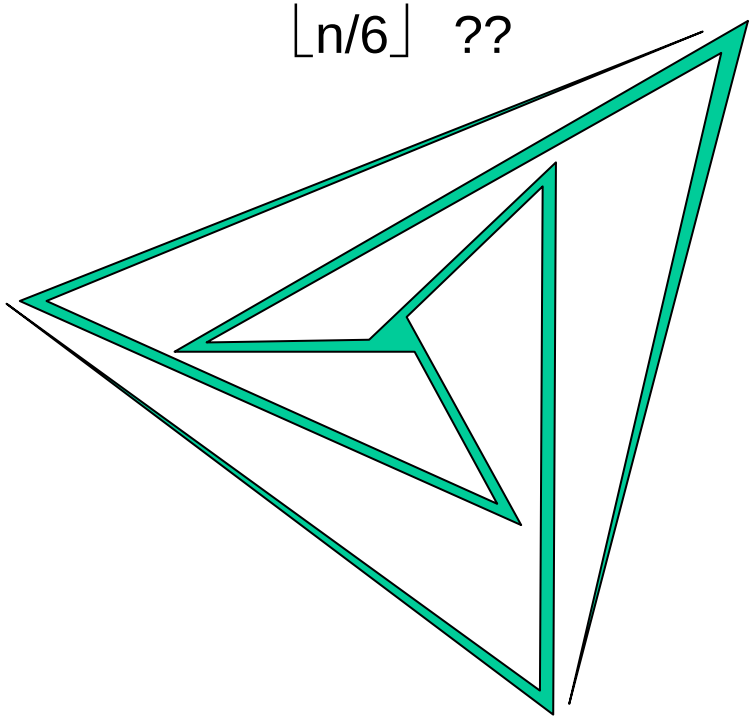
Urrutia, 06

VISIBILITY THROUGH THE WALLS

COMBINATORIAL PROBLEM

2-routers

$\lfloor n/6 \rfloor$??



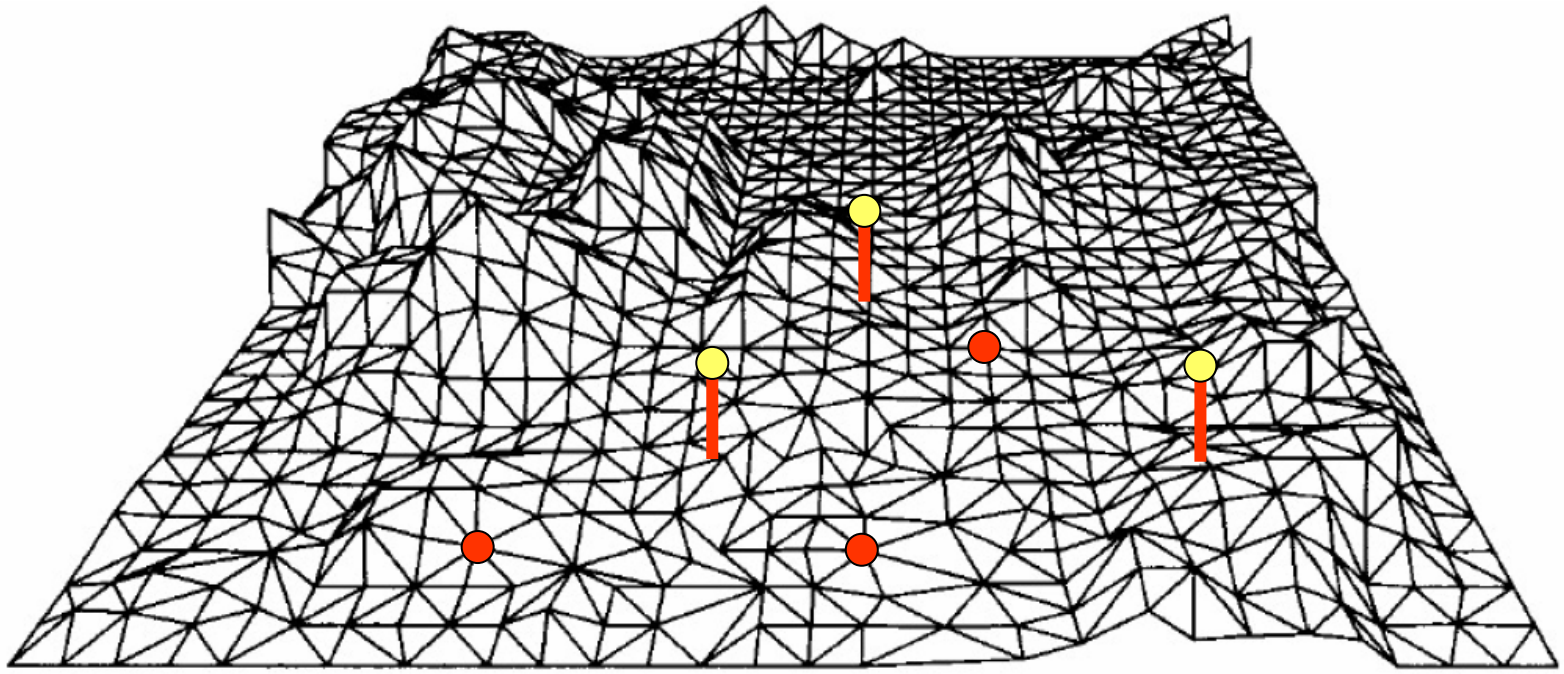
Wireless router!!

ALGORITHMIC PROBLEM

OPEN PROBLEM

TERRAIN GUARDING

How many guards?



Minimize is a NP-hard problem
Cole-Sharir, 89

VERTEX (POINT) GUARD
FIXED HEIGHT GUARD

TERRAIN GUARDING

How many guards? **PLANE TRIANGULATIONS**

Vertex-guards

$\lfloor n/2 \rfloor$ are always sufficient and sometimes necessary
Bose, Shermer, Toussaint, Zhu, 92

Edge-guards

$\lfloor n/3 \rfloor$ are always sufficient (Everett, Rivera-Campo, 94) and
$\lfloor (4n-4)/13 \rfloor$ are sometimes necessary (BSTZ, 92, 97)

TERRAIN GUARDING

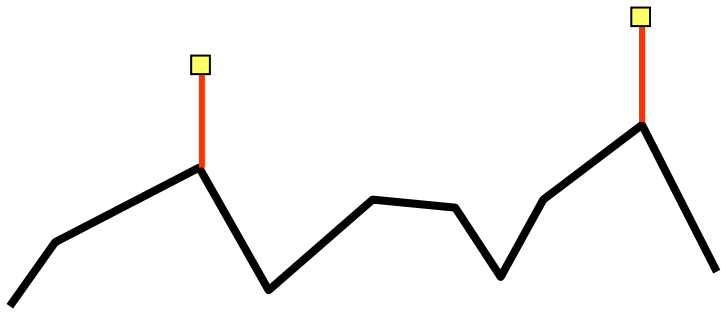
Watchtower placement problems

- Discrete version — bases at vertices of T
- Continous version — bases anywhere on T

There are k watchtowers, we want to minimize their height

- Single watchtower — O(nlogn) Zhu, 97

- Two watchtowers — Agarwal, Bereg, Ntafos, Zhu, 05
  - 1.5D discrete version — $O(n^2 \log^4 n)$
  - continous version — $O(n^3 \alpha(n) \log^3 n)$



TERRAIN GUARDING

Watchtower placement problems

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  - 1.5D discrete version $O(n^2 \log^4 n)$
  - continous version $O(n^3 \alpha(n) \log^3 n)$
  - 2.5D discrete version $O(n^{11/3} \text{polylog}(n))$

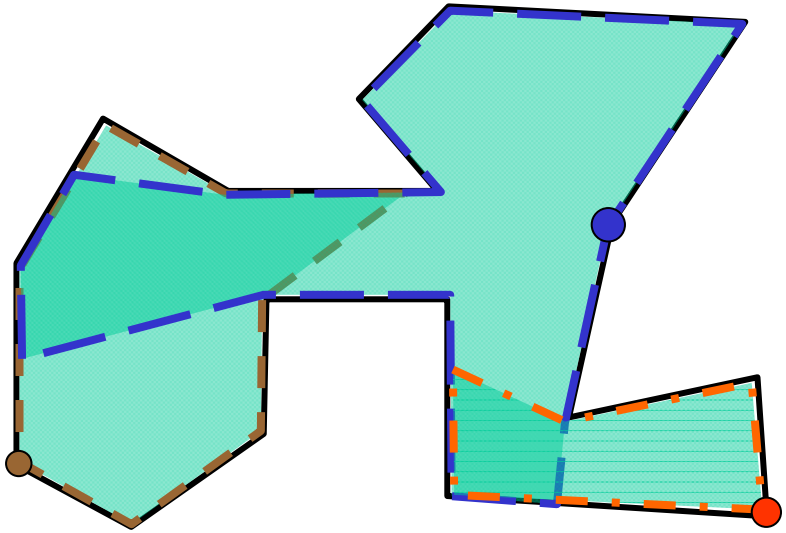
ART GALLERY Problem

APPROXIMATION ALGORITHMS

An algorithm A for solving an minimization problem with cost function f, is a s-approximation algorithm if

$$f(\text{solution A}) \leq s \cdot f(\text{optimum solution})$$

s factor of approximation



Reduction to **SET COVER**

Visibility polygons of the vertices

ART GALLERY Problem

APPROXIMATION ALGORITHMS

Logarithmic approximations

Ghosh, 87

The greedy algorithm for **SET COVER** produces a vertex-guard cover

O(logn)-approximation time $O(n^5 \log n)$

Efrat, Har-Peled, 02 (using randomization and VC-dimension)

- vertex-guard cover

  If c is the optimum

  O(log c)-approximation time $O(nc^2 \log^4 n)$

- guards in a grid

  O(log c)-approximation time $O(nc^2 \log c \log(nc) \log^2 \Delta)$

No approximation bounds for the greedy approach are known if guards can be located in the interior of the polygon

ART GALLERY Problem

APPROXIMATION ALGORITHMS

Constant-factor approximations

Nilsson, 05

- monotone polygons (interior guards)
  12-approximation polynomial time
- orthogonal polygon (interior guards)
  96c-approximation

1.5D terrains (exploiting geometric structure of terrains)

- Ben-Moshe, Katz, Mitchell, 04
  O(1)-approximation time $O(n^4)$
- King, 06
  5-approximation time $O(n^2)$ discrete and continuos

ART GALLERY Problem

APPROXIMATION ALGORITHMS

(In-)Approximability, Eidenbenz, 2000

- Polygons without holes

  MINIMUM VERTEX (POINT) GUARD is APX-hard

  There exists $\varepsilon>0$ such that no polynomial time algorithm can achieve an approximation ratio of $1+\varepsilon$

- Polygons with holes

  MINIMUM VERTEX (POINT) GUARD can not be approximated with an approximation ratio of O(logn)

The problem is O(logn)-complete for vertex-guard

APPROXIMATION ALGORITHMS

ART GALLERY Problem

A ONE POINT GUARD

B ONE POINT GUARD (holes)

C MINIMUM VERTEX/EDGE GUARD

D MINIMUM VERTEX/EDGE GUARD (holes)

E MINIMUM POINT GUARD

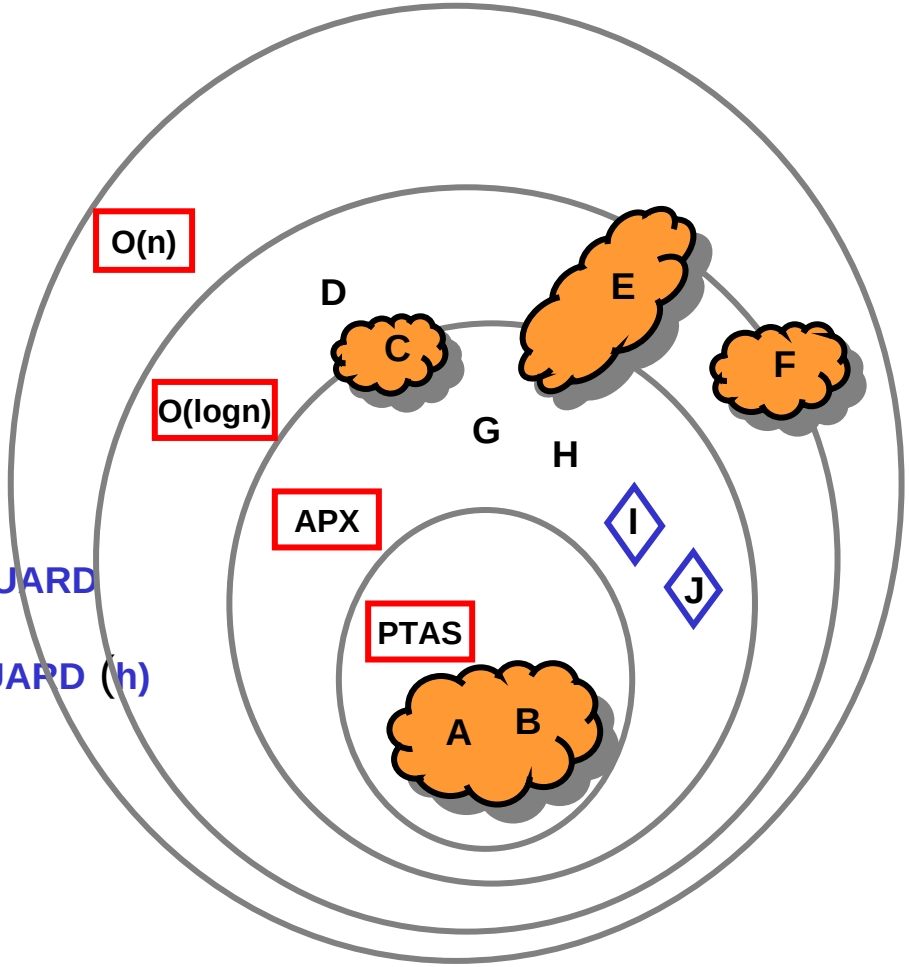
F MINIMUM POINT GUARD (holes)

G MAX LENGTH/AREA/VALUE VERTEX/EDGE GUARD

H MAX LENGTH/AREA/VALUE VERTEX/EDGE GUARD (h)

I MAX LENGTH/AREA/VALUE POINT GUARD

J MAX LENGTH/AREA/VALUE POINT GUARD (h)



APPROXIMATION ALGORITHMS

ART GALLERY Problem

- Bajuelos, Marques, Martins, Tomás, 04, 05, 06

  Vertex Guard problem for orthogonal polygons

- Bottino, Laurentini, 04, 05, 06

  Optimal positioning of sensors in 2D
  Integer Edge Covering in polygons

- Amit, Mitchell, Packer, ’07

  Systematic experimentation with many guard placement heuristics

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- Canales, Abellanas, Alba, H., 04, 07

  Point Guard problem

- Bajuelos, Canales, H., Martins, 07, 08

  Vertex Guard problem
  Maximum Hidden Vertex Set problem

Heuristics
Genetic algorithm,
Simulated annealing

ART GALLERY Problem

APPROXIMATION ALGORITHMS

An algorithm A for solving an minimization problem with cost function f, is a s-approximation algorithm if

$$f(\text{solution A}) \leq s \cdot f(\text{optimum solution})$$

s factor of approximation

.... but we don't know the optimum solution!

Basic references for Art Gallery problems

- J. Urrutia, *Art gallery and illumination problems,* Handbook on Computational Geometry, (J.-R. Sack, J. Urrutia, ed.) Elsevier, 2000
- J. O'Rourke, *Art Gallery, Theorems and Algorithms,* Oxford, 1987

Open problems

- The Open Problems Project (Demaine, Mitchell, O'Rourke) http://maven.smith.edu/~orourke/TOPP/