ON THE LACK OF EQUIDISTRIBUTION ON FAT CANTOR SETS

GABRIEL FUHRMANN

Given an irrational rotation number $\omega \in \mathbb{R} \setminus \mathbb{Q}$, a set $W \subseteq \mathbb{T}^1 (= \mathbb{R}/\mathbb{Z})$ and a point $x \in \mathbb{T}^1$, we define

$$S_W^n(x) = 1/n \cdot \sum_{\ell=0}^{n-1} \mathbf{1}_W(x + \ell\omega) \qquad (n \in \mathbb{N}),$$

where $\mathbf{1}_W$ denotes the characteristic function of W.

It is well-known and straightforward to see that if W is a Cantor set, then there is a dense (in fact, residual) set of $x \in \mathbb{T}^1$ with $\lim_{n\to\infty} S_W^n(x) = 0$. On the other hand, if W is a *fat* Cantor set (that is, $\mathrm{Leb}_{\mathbb{T}^1}(W) > 0$), we have $\lim_{n\to\infty} S_W^n(x) = \mathrm{Leb}_{\mathbb{T}^1}(W) > 0$ for $\mathrm{Leb}_{\mathbb{T}^1}$-a.e. x.

But what other frequencies of visits to W may occur? In the words of a recent MathOverflow post [1], what is the set

$$S_W = \bigcup_{x\in\mathbb{T}^1} \bigcap_{N\in\mathbb{N}} \overline{\{S_W^n(x) \colon n \geq N\}}?$$

We give a first answer to the above question by showing that every irrational rotation allows for certain fat Cantor sets C such that S_C is maximal, that is, $S_C = [0, \mathrm{Leb}_{\mathbb{T}^1}(C)]$.

In this talk, I will focus on discussing some of the basic tools behind the above result.

References

[1] D. Kwietniak, *Possible Birkhoff spectra for irrational rotations*, MathOverflow (2020), https://mathoverflow.net/q/355860 (version: 2020-03-27).