On basic concepts, kinematics and governing equations in integral form in Fluid Mechanics

E. T. S. Ingeniería y Diseño Industrial
UNIVERSIDAD POLITÉCNICA DE MADRID

FRANCISCO CRUZ MAZO
https://blogs.upm.es/fcruzmazo

These supplementary notes are intended to help any student better understand the theoretical explanations given by the professor on the blackboard. Please note that these notes are not a substitute for active class attendance.

1 Basic concepts

Definition of fluid - The substantial difference that distinguishes a fluid from a solid is that, whether in its liquid or gaseous form, a fluid does not resist deformation but does resist the rate of deformation, leading to the concept of viscosity, μ $[Pa{\cdot}s]$, which will be elaborated upon later. However, this definition is vague since there are solids that can exhibit fluid-like behaviors and vice versa, which we will not address in this introductory course.

Fluid as a continuum medium - We treat the fluid as a continuous medium, and therefore its dynamics can be described as functions for each position $\vec{r}$ and time t with respect to a certain reference system. The continuum medium hypothesis is based on the idea that the volume of the minimum unit of fluid, the fluid particle, contains a number, N, in a volume δV, sufficient individual molecules of mass m_i, length d_o, and velocity $\vec{w}_i$, such that average measures of density ρ and average velocity $\vec{v}$ can be defined:

$$\rho(\vec{r},t) = \lim_{\delta V \to 0} \frac{\sum_{i=1}^{N(\delta V)} m_i}{\delta V} \qquad \left[\frac{kg}{m^3}\right] \tag{1}$$

$$\vec{v}(\vec{r},t) = \lim_{\delta V \to 0} \frac{\sum_{i=1}^{N(\delta V)} m_i \vec{w}_i}{\sum_{i=1}^{N(\delta V)} m_i} \qquad \left[\frac{m}{s}\right] \tag{2}$$

Local thermodynamic equilibrium The length $\delta V^{1/3}$ must also satisfy that it is sufficiently large compared to the mean free path λ of molecular collisions so that local conditions of local reversibility or local thermodynamic equilibrium can be assumed. Thus, the Knudsen number $Kn = \lambda/L \ll 1$. In this way, we can apply the equations of state to the volume of the fluid particle, which in turn defines the relationship between intensive variables. This is the substantial difference from Thermodynamics, which assumes these equilibrium states at large finite extensive volumes, and that in Fluid Mechanics, we relax to its infinitesimal version. Two of those intensive quantities are pressure p $[Pa]$ and total energy per unit mass ϵ [J/kg]. On one hand, pressure p arises from the differential force $d\vec{F}$ projected onto the normal direction $\vec{n}$ and outward direction of a differential surface dS, with the expression:

$$p(\vec{r},t) = -\frac{d\vec{F}\cdot\vec{n}}{dS} = -\frac{dF_n}{dS} \qquad [Pa] \tag{3}$$

where F_n is the normal component of the force. Regarding total energy per unit mass, ϵ has two components: a kinetic component $|\vec{v}|^2/2$ associated with the instantaneous translational motion of the fluid particle, and an internal component $e = e_c + e_p$, which in turn has two sub-components: the average molecular kinetic agitation e_c and another in the potential e_p of intermolecular forces that decay significantly with intermolecular distance d, being much weaker in gases ($d \gg d_o$) than in liquids ($d \sim d_o$). Thus

$$\epsilon = \frac{1}{2}|\vec{v}|^2 + e \qquad [J/kg] \tag{4}$$

The average temperature $T(\vec{r},t)$ in the fluid particle is somewhat related to e_c through the universal Boltzmann constant $k_B = 1.38\cdot 10^{-23}$ J/K with the kinetic energy of agitation of the average molecular mass m_c, $m_c c^2 = k_B T$, where c is the average molecular agitation velocity. Regarding local reversibility, the equation for entropy S per unit mass allows it to be related to its internal energy counterpart and the set of thermodynamic variables: pressure, density, and temperature, such that

$$TdS = de + pd\left(\frac{1}{\rho}\right) \tag{5}$$

and other thermodynamic variables of interest such as the enthalpy per unit mass $h = e + p/\rho$, the specific heats at constant pressure $c_p = T(\partial S/\partial T)_p$, at constant volume $c_v = T(\partial S/\partial T)_v$, the specific heat ratio $\gamma = c_p/c_v$, the compressibility coefficient $\kappa = -\rho^{-1}(\partial\rho/\partial p)_T$, the thermal expansion coefficient $\beta = -\rho^{-1}(\partial\rho/\partial T)_p$, and the speed of sound $a = (\partial p/\partial\rho)_S$. For practical purposes, the interrelation between the different intensive quantities is articulated through the equation of state. In our case, we consider the cases of ideal gas (i.g.) and perfect gases:

$$\frac{p^{(g.i.)}}{\rho^{(g.i.)}} = R_g T^{(g.i.)}, \tag{6}$$

where R_g is the gas constant and depends on the universal gas constant $R = 8.314$ $J/mol\cdot K$ and the molecular mass of the gas, M, such that $R_g = R/M$. For an ideal gas, the internal energy is only a function of temperature such that $de = c_v dT$, with c_v constant, as well as the enthalpy such that $dh = c_p dT$ with c_p also constant. In fact, $\gamma = c_p/c_v = 1 + R_g/c_v$. Thus, the reversible entropy equation results in $TdS = c_v dT - R_g T d\rho/\rho$, which for an isentropic process ($dS = 0$) starting from an initial state 0, yields:

$$\frac{p^{(isen)}}{\rho^{\gamma(isen)}} = \frac{p_o^{(isen)}}{\rho_o^{\gamma(isen)}} \tag{7}$$

For a perfect liquid (p.l.), everything becomes simpler as its equation of state:

$$\rho^{(l.p.)} = constant, \tag{8}$$

and its internal energy also depends only on the temperature $de = cdT$, with $c = c_p = c_v$ as specific heat equal to constant pressure and volume such that $\gamma = 1$, with differential entropy $TdS = de = cdT$, so that for a certain isentropic process the temperature remains unchanged.

Overview of fluid phenomena - The are key features of fluids that deserve attention given their importance in science and technology.

Viscosity - μ $[Pa \cdot s]$ is related to the diffusive transport of momentum between the fluid layers, transmitting a tangential shear stress τ between the layers, which is a function of the tangential deformation rate $\dot{\varepsilon}$, varying the parallel and tangential velocity v_t according to the normal coordinate n. This issue will be elaborated on in depth in later topics, but, preliminarily, this particular expression can be indicated under these conditions:

$$\tau = \mu \frac{\partial v_t}{\partial n} = \mu \dot{\varepsilon}, \tag{9}$$

where the viscosity μ is constant and thus corresponds to a Newtonian fluid. There are non-Newtonian fluids whose viscosity may apparently decrease with the deformation rate,*shear thinning*, or increase with it,*shear thickening*. Note how the viscosity of liquids decreases with the increase in temperature, in contrast to what occurs with gases. The Reynolds number is a dimensionless number defined by Re= $\rho v L/\mu$, for a fluid of characteristic density ρ and viscosity μ, which flows at a characteristic velocity v through a characteristic length L.

Vapor pressure and cavitation - Water responds to equilibrium situations that, depending on pressure and temperature, manifest themselves as liquid, solid, or gaseous phases. The vapor pressure, $p_v(T)$, at a certain temperature T represents the pressure under which water transitions from the liquid phase to the gaseous phase, that is, it cavitates. This phenomenon is typically undesirable in hydraulic systems, where the abrupt proliferation of bubbles can be very detrimental to various components, from pipes to hydraulic machines. The vapor pressure increases with an increase in temperature.

Equilibrium and surface tension - At interfaces between immiscible fluids, a very interesting phenomenon occurs: a continuous and bidirectional exchange of molecules in a very narrow region called the interface. From the balance of forces at this interface and averaging the intermolecular effects, a property can be measured: the surface tension σ $[N/m]$, which relates the pressure jump across it as a function of its local curvature radii. This is the Laplace-Young law in its friendly version:

$$p_1 - p_2 = \sigma \left(\frac{1}{R_i} + \frac{1}{R_j} \right), \tag{10}$$

where p_1 and p_2 represent the pressure on the concave and convex sides, respectively. For example, for a drop of radius R, $R_i = R_j = R$ leading to $p_1^{(drop)} - p_2^{(outside)} = 2\sigma/R$, while for a bubble (a bubble within a drop), it would be $p_1^{(bubble)} - p_2^{(outside)} = 4\sigma/R$, and for a jet of radius R, it would be $p_1^{(jet)} - p_2^{(outside)} = \sigma/R$. If the interface is not free but is in contact with a solid, a local equilibrium is formed with it, manifested macroscopically through a contact angle θ_c. A classic example is the ascent of liquid in a capillary tube with respect to the height of the level of the bath in which it is partially submerged. Another classic example is a droplet that rests on a solid surface that can exhibit hydrophilic behavior ($\theta_c < \pi/2$) or hydrophobic behavior ($\theta_c >$ \o” $\gg \pi/2$). The dimensionless Bond number $Bo = \rho g L^2/\sigma$ allows weighing the relative contribution of stresses due to surface tension with respect to variations in hydrostatic pressure from the gravitational field as a function of the characteristic size L.

2 Kinematics of the fluid particle

Velocity and vorticity fields - Once the fluid particle is defined as the minimum unit, we proceed to characterize its movement. Specifically, there are two ways to do this: following a Lagrangian approach to describe the velocity field ($\vec{v}^{(lag)}(\vec{r}(t))$ with an identification for each fluid particle), or with an Eulerian approach where space and time are separated as independent variables on which the velocity field ($\vec{v}^{(eul)}(\vec{r}, t)$) depends, and not identified with the evolution of a specific fluid particle. Another vector field used to describe the movement is the vorticity, $\vec{\omega} = \nabla \times \vec{v}$, as the rotational velocity, and the angular velocity $\vec{\Omega}$ with the following relationship between them: $\vec{\omega} = 2\vec{\Omega}$

Types of fluid flows - Flows can be classified according to the functional dependence of the velocity field $\vec{v}$ and vorticity $\vec{\omega}$:

- Space: number of spatial coordinates on which $\vec{v}$ depends, which can be one-dimensional (1), two-dimensional (2), and three-dimensional (3).
- Time: if $\vec{v}$ does not depend on time, it is stationary; and if it has temporal dependence, it is nonstationary or transient.
- Direction: It is a unidirectional flow if $\vec{v}$ has only one component in a uniform direction with respect to a reference system, being bidirectional or tridirectional if it has two or three components, respectively.
- Vorticity: an irrotational flow has zero vorticity, being rotational otherwise.

Pathline, streamline and streakline - Regarding the motion associated with a certain velocity field of a flow, information can be extracted about different elements:

- Pathline: trajectory $\vec{r}(t)$ of a fluid particle over time given $\frac{d\vec{r}}{dt} = \vec{v}$ at $\vec{r}(t = t_o) = \vec{r}_o$.
- Steady path: steady version of the trajectory as a fluid road that is revealed when removing the temporal parameter from the pathline equations $\vec{r}$.
- Streamline: a line where the fluid velocity field is tangent $\vec{v} \times d\vec{r} = 0$.

- Streakline: a line that connects all the fluid particles that ever passed a certain spatial point.

Material derivative. Local and convective acceleration - The evolution of a certain variable ϕ following the motion of a fluid particle considers the independence of space and time in the Eulerian approach, which requires us to evaluate this evolution in terms of the total differential, with this expression following the chain rule $d\phi = dt\frac{\partial\phi}{\partial t} + d\vec{r}\cdot\nabla\phi$. Dividing by a nonzero dt results in the material derivative ($D\phi/Dt$) of the variable ϕ, which can, without loss of generality, be a scalar, vector, or tensor quantity.

$$\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + (\vec{v}\cdot\nabla)\phi \tag{11}$$

This quantity ϕ, when identified with the velocity field $\phi = \vec{v}$, provides us with the material or substantial derivative of the velocity, the acceleration $\vec{a}$, such that:

$$\vec{a} = \frac{D\vec{v}}{Dt} = \frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}, \tag{12}$$

where the local and convective accelerations are the first and second terms of the second member, respectively. On the other hand, the convective acceleration can be decomposed into a part of modular variation and another of directional variation as the first and second terms of the second member of the following expression, respectively:

$$\vec{v}\cdot\nabla\vec{v} = \nabla\left(\frac{|\vec{v}|^2}{2}\right) - \vec{v}\times(\nabla\times\vec{v}) \tag{13}$$

Gradient of velocity - It is interesting, as will be explained in the following sections, to characterize the movement around a fluid particle between two moments separated by a dt. We evaluated the differential of the velocity field $\vec{v}$ and its spatial relationship using the chain rule without apparent temporal variation.

$$d\vec{v} = d\vec{r}\cdot\nabla\vec{v} = d\vec{r}\cdot\left[\frac{1}{2}\left(\nabla\vec{v}+\nabla\vec{v}^T\right) + \frac{1}{2}\left(\nabla\vec{v}-\nabla\vec{v}^T\right)\right] = d\vec{r}\cdot\bar{\bar{\varepsilon}} + d\vec{r}\cdot\bar{\bar{\gamma}}, \tag{14}$$

where $\nabla\vec{v}$ is the gradient of velocity , a tensor that can be decomposed into the sum of $\bar{\bar{\varepsilon}}$, the strain rate tensor (symmetric), and $\bar{\bar{\gamma}}$, the rotation rate tensor (antisymmetric). The former is associated with longitudinal and angular strain rates on one side ($d\vec{v}_{def} = d\vec{r}\cdot\bar{\bar{\varepsilon}}$), and the latter relates to the rigid body rotation velocity ($d\vec{v}_{rot} = d\vec{r}\cdot\bar{\bar{\gamma}} = \frac{1}{2}\vec{\omega}\times d\vec{r}$), respectively.

As an example, the strain rate tensor is developed for a velocity field in the following way in Cartesian coordinates x, y, z with unit vectors. $\vec{i}, \vec{j}, \vec{k}$ tal que $\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$:

$$\begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{1}{2}\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial v_x}{\partial z}+\frac{\partial v_z}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right) & \frac{\partial v_y}{\partial y} & \frac{1}{2}\left(\frac{\partial v_y}{\partial z}+\frac{\partial v_z}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial v_x}{\partial z}+\frac{\partial v_z}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial v_y}{\partial z}+\frac{\partial v_z}{\partial y}\right) & \frac{\partial v_z}{\partial z} \end{bmatrix} \tag{15}$$

Similarly, in Cartesian coordinates, the rotation velocity tensor is::

$$\begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) \\ -\frac{1}{2}\left(\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x}\right) & 0 & \frac{1}{2}\left(\frac{\partial v_y}{\partial z} - \frac{\partial v_z}{\partial y}\right) \\ -\frac{1}{2}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) & -\frac{1}{2}\left(\frac{\partial v_y}{\partial z} - \frac{\partial v_z}{\partial y}\right) & 0 \end{bmatrix} \tag{16}$$

Convective Flow and Reynolds Transport Theorem - At this point, we will introduce the concept of flow Φ_S of a generic quantity ϕ that is convected with a velocity $\vec{v}$ through a surface S with a normal vector $\vec{n}$ given by the following expression:

$$\Phi_S = \int_S \phi \left(\vec{v} \cdot \vec{n}\right) dS \tag{17}$$

The total variation over time of a quantity ϕ within a fluid volume, $V(t^*)$, at a parametric time t^* can be related to its local and convective variations according to a control volume that varies over time at t^*, according to the Reynolds Transport Theorem, with the following evolution:

$$\frac{d}{dt}\int_{V_f(t^*)} \phi dV = \frac{d}{dt}\int_{V_c(t^*)} \phi dV + \oint_{S_c(t^*)} \phi \left(\vec{v} - \vec{v}_c\right) \cdot \vec{n} dS \tag{18}$$

Similarly, the version of Reynolds' transport Theorem for a control volume such as fluid volume $V(t)$, whose velocity is that of the fluid such that $\vec{v}_c = \vec{v}$, takes the following form:

$$\frac{d}{dt}\int_{V_f(t)} \phi dV = \int_{V_f(t)} \frac{\partial \phi}{\partial t} dV + \oint_{S_c(t)} \phi \left(\vec{v} \cdot \vec{n}\right) dS \tag{19}$$

3 Integral equations in fluid mechanics

Integral equation for mass conservation - The Reynolds transport Theorem, in the case where the quantity is the density $ho = ho$, provides us with the integral form of the mass conservation equation, considering that the temporal variation of the mass contained in the fluid volume, as follows, is zero: $racddt \int_{V_f(t)} \rho dV = 0$, thus resulting in:

$$0 = \frac{d}{dt}\int_{V_f(t^*)} \rho dV = \frac{d}{dt}\int_{V_c(t^*)} \rho dV + \oint_{S_c(t^*)} \rho \left(\vec{v} - \vec{v}_c\right) \cdot \vec{n} dS \tag{20}$$

Integral equations for momentum conservation - *Volumetric forces* - In this course, we focus on the forces that act on a differential volume due to its mass and inertia. Thus, the sum of forces per unit volume originates from the gravitational field and the noninertiality of the reference system with acceleration ($\vec{a}_o$) and angular velocity ($\vec{\Omega}_o$) on which the velocity field ($\vec{v}$) is expressed. In the general case, we recall the expression from previous courses:

$$\vec{f_v} = \rho\vec{g} - \rho\left(\vec{a}_o + \vec{\Omega}_o \times (\vec{\Omega}_o \times \vec{r}) + 2\vec{\Omega}_o \times \vec{v} + \frac{d\vec{\Omega}_o}{dt} \times \vec{r}\right) \tag{21}$$

Surface forces and stress tensor - The treatment of surface forces, ($\vec{f_s}$), deserves special attention because at this point we refer to the stress tensor ($\bar{\bar{\tau}}$) of the surface forces:

$$\vec{f_s} = \bar{\bar{\tau}} \cdot \vec{n}, \tag{22}$$

Where ($\bar{\bar{\tau}}$), a second-order tensor, can be decomposed according to its contributions from static pressure (p) and viscous stresses ($\bar{\bar{\tau}}_v$):

$$\bar{\bar{\tau}} = -p\bar{\bar{I}} + \bar{\bar{\tau}}_v, \tag{23}$$

with (I) as the identity matrix. The modeling of ($\bar{\bar{\tau}}_v$) is performed through the Navier-Poisson constitutive law, relating linear form to the viscosity (μ) applied to the deformation rate tensor ($\bar{\bar{\varepsilon}}$) such that:

$$\bar{\bar{\tau}}_v = 2\mu\bar{\bar{\varepsilon}} + \left(\mu_v - \frac{2}{3}\mu\right)(\nabla \cdot \vec{v})\,\bar{\bar{I}}, \tag{24}$$

where (μ_v) is the volumetric viscosity coefficient related to the degrees of freedom of gases, and in the case of incompressible fluids, since ($\nabla \cdot v = 0$), the second term would be zero.

If at this point we specify the linear moment per unit volume ($\phi = \rho\vec{v}$), we will obtain the local and convective variation of the *linear momentum* that must be balanced with the sum of the previous volume forces ($\vec{f_v}$) and surface forces ($\vec{f_s}$) acting on the control volume:

$$\begin{aligned}\frac{d}{dt}\int_{V_f(t^*)} \rho\vec{v}dV = \frac{d}{dt}\int_{V_c(t^*)} \rho\vec{v}dV + \oint_{S_c(t^*)} \rho\vec{v}\left(\vec{v} - \vec{v}_c\right) \cdot \vec{n}dS = \\ = \int_{V_c(t^*)} \vec{f_v}dV + \oint_{S_c(t^*)} \left(-p\bar{\bar{I}} + \bar{\bar{\tau}}_v\right) \cdot \vec{n}dS\end{aligned} \tag{25}$$

In the particular case of obtaining the conservation equation for *angular momentum*, we impose that ($\phi = \rho\vec{v} \times \vec{r}$) and obtain the integral equation by virtue of Reynolds' Transport Theorem:

$$\begin{aligned}\frac{d}{dt}\int_{V_f(t^*)} \vec{r} \times \rho\vec{v}dV = \frac{d}{dt}\int_{V_c(t^*)} \vec{r} \times \rho\vec{v}dV + \oint_{S_c(t^*)} \vec{r} \times \rho\vec{v}\left(\vec{v} - \vec{v}_c\right) \cdot \vec{n}dS = \\ = \int_{V_c(t^*)} \vec{r} \times \vec{f_v}dV + \oint_{S_c(t^*)} \vec{r} \times \left(-p\bar{\bar{I}} + \bar{\bar{\tau}}_v\right) \cdot \vec{n}dS\end{aligned} \tag{26}$$

Integral equation for energy conservation - Similarly, Reynolds' Transport Theorem can be applied to the total energy per unit volume ($\rho\left(e + \frac{1}{2}v^2\right)$), whose space-time evolution will be balanced by the net flux of heat power per unit area ($\vec{q}$) transferred from the environment with the power developed by the *real* volumetric and surface forces, as well as sources or sinks of heat power through physicochemical mechanisms such as radiation or chemical reactions, which we will skip in this introductory course. Thus, the integral equation of energy is:

$$\begin{gathered}\frac{d}{dt}\int_{V_f(t^*)}\rho\left(e+\frac{1}{2}v^2\right)dV=\frac{d}{dt}\int_{V_c(t^*)}\rho\left(e+\frac{1}{2}v^2\right)dV+\\ \oint_{S_c(t^*)}\rho\left(e+\frac{1}{2}v^2\right)(\vec{v}-\vec{v}_c)\cdot\vec{n}dS=\int_{V_c(t^*)}\rho\vec{g}\cdot\vec{v}dV+\\ +\oint_{S_c(t^*)}-\vec{v}\cdot p\bar{\bar{I}}\cdot\vec{n}dS+\oint_{S_c(t^*)}\vec{v}\cdot\bar{\bar{\tau}}_v\cdot\vec{n}dS+\oint_{S_c(t^*)}\vec{q}\cdot\vec{n}dS,\end{gathered} \tag{27}$$

where the Rayleigh viscous dissipation term can be demonstrated to be greater than or equal to zero in order for the energy conservation equation to define a realizable process that respects the second law of Thermodynamics. Regarding ($\vec{q}$), the most commonly used constitutive law is Fourier's law for conduction, which relates the temperature gradient (∇T) to a property, the thermal conductivity of the fluid (k) ($[W/(mK)]$), such that ($\vec{q}=-k\nabla T$). It should be noted here that the misnamed heat convection is, in short, the result of heat transport by conduction through coupled thermo-fluid dynamic flow.