UNIVERSITÀ DEGLI STUDI DI BARI

Facoltà di Scienze Matematiche, Fisiche e Naturali

Corso di Laurea Magistrale in Matematica

Tesi di Laurea in Analisi Matematica

EQUAZIONI LINEARI E NON LINEARI DELLE ONDE CON DAMPING

Linear and non-linear damped wave equations

Relatore:

Chiar.ma Prof. Sandra Lucente;

Chiar.mo Prof. Michael Reissig.

Laureando:

Fabio Pizzichillo

Anno Accademico 2013-2014

Contents

Indice

Introduction

The study of waves constitutes one of most important research field of mathematics and physics. One can find this kind of model in description of the study of propagation of sound and light, in the analysis of sea waves, in the transmission of signals or in earthquakes.

In this thesis we study some problems related to linear and semi-linear wave equations with some damping terms: Klein-Gordon equation with a classical damping term and wave equation with a visco-elastic damping term. We can use these equation to describe some natural phenomena. For example, if we study the propagation of sound in different media, then we have an external damping. If we study the movement of some visco-elastic material, then a structural or visco-elastic damping appears.

To threat both cases the first step is the study of linear problems. Applying partial Fourier transformation one gets a representation formula of the solution in the phase space. With this strategy we obtain results about well-posedness, propagation of singularities and behaviour of classical energy and energies of higher order as well.

An important tool to study both models is the definition of a suitable energy. For the damped Klein-Gordon model we choose the classical Klein-Gordon energy. It consists of the elastic, kinetic and potential energy. For visco-elastic damped models we choose the classical wave energy, it means, the kinetic and elastic energy. We estimate these quantities and, consequently, we use these estimates to study semi-linear model.

Our semi-linear models contain sources of different types of power non-linearity. As power non-linearity we consider $|u|^p$, $|u_t|^p$ and $||D|^a|^p$ for the visco-elastic damped wave model and $|u|^p$ for the damped Klein-Gordon model. In particular $||D|^a|^p$ is a pseudo-differential non-linearity so it is not local.

Local existence in time is more or less clear even for large data. More interesting is the question about globally in time small data solutions. Such results could be used for proving stability results of the zero solution as steady state solution.

The strategy concerns in to define a suitable function space and to define on this space an operator N such that the solution we are trying to find is the unique fixed point of N. For this reason we use Banach's fixed point theorem. The operator N is defined in integral form by using Duhamel's principle. To prove that N is a contraction mapping, we have to estimate the non-linear term. For this purpose we will use some inequalities as Gagliardo-Nirenberg inequality in a generalized form which we get from [6], and some superposition results from [8].

This thesis is organized as follows:

In the first chapter we study the damped Klein-Gordon model. We estimate the energy and obtain exponential decay if we suppose L^2 regularity of initial data.

In Chapter 2 we study the visco-elastic damped wave model. We observe that only L^2 regularity for the data is not sufficient to obtain decay. So we ask for additional re-gularity, it means $L^2 \cap L^1$ regularity. Under these assumptions we study decay estimates even for higher order derivatives of the solution. We obtain potential decay.

In Chapter 3 we study visco-elastic damped wave models with power non-linearity $|u|^p$. Results about these model are already known by [2]. In order to generalize such results in chapters 4,5 and 6, we prove here global existence result for admissible exponents and small initial data in $(H^2 \cap L^1) \times (L^2 \cap L^1)$.

In Chapter 4 we study damped Klein-Gordon models with power non-linearity

$|u|^p$. We obtain globally (in time) existence of small data solutions for all exponents $p > 1$. For this model it is sufficient to suppose initial data in $H^1 \times L^2$ because of the exponential decay.

In Chapter 5 we focus our attention on visco-elastic damped wave models with power non-linearity $|u_t|^p$. We ask for $(H^s \cap L^1) \times (H^{s-2} \cap L^1)$ regularity of initial data with a sufficiently large s. Under these assumptions we prove global (in time) existence of small data solutions for all exponents $p > s$.

Finally, Chapter 6 is devoted to visco-elastic damped wave models with power non-linearity $||D|^a|^p$, where $a \in (0, 2)$. First we ask for $(H^2 \cap L^1) \times (L^2 \cap L^1)$ regularity of initial data. With this regularity we obtain global existence of solutions but with some restrictions to a and to the dimension n. So we ask more regularity for initial data, it means $(H^s \cap L^1) \times (H^{s-2} \cap L^1)$ with s large. In this way we find global solutions for small initial data for all exponents $p > s$.

Introduzione

Lo studio delle onde rappresenta uno dei più importanti campi di ricerca nella matematica e nella fisica. Si possono trovare questo tipo di modelli nello studio della propagazione del suono e della luce, nell'analisi delle onde marine, nella trasmissione dei segnali o nella descrizione dei terremoti.

In questa tesi abbiamo studio alcuni problemi lineari e semi-lineari relativi all'equazione delle onde con certi termini di smorzamento: l'equazione di Klein-Gordon con termine di smorzamento classico e l'equazione delle onde con termine di smorzamento visco-elastico. Anche queste equazioni descrivono fenomeni naturali: per esempio se studiamo la propagazione del suono in mezzi differenti abbiamo uno smorzamento classico, se studiamo il movimento di alcuni materiali visco-elastici allora appare uno smorzamento strutturale o visco-elastico.

Per trattare entrambi i casi il primo passo è lo studio dei problemi lineari. Applicando la trasformata di Fourier si ottiene una formula di rappresentazione della soluzione nello spazio delle fasi. Con questa strategia abbiamo risultati di buona positura, propagazione delle singolarità e comportamento dell'energia classica e di ordine superiore,

Uno strumento importante per studiare entrambi i modelli è definire opportunamente l'energia. Per il modello Klein-Gordon con smorzamento classico abbiamo la stessa energia del modello classico Klein-Gordon cioè energia elastica, cinetica e potenziale. Per il modello con smorzamento visco-elastico abbiamo definito l'energia nello stesso modo del modello classico delle onde ovvero energia cinetica ed elastica. Abbiamo stimato queste quantità e abbiamo utilizzato queste stime nello studio del modello semi-lineare.

I modelli semi-lineari contengono tipi differenti di non-linearità. Abbiamo considerato le non-linearità di tipo $|u|^p$, $|u_t|^p$ e $||D|^a u|^p$ per l'equazione delle onde con smorzamento visco-elastico e $|u|^p$ per l'equazione di Klein-Gordon con smorzamento classico. In particolare $||D|^a u|^p$ è una non-linearità pseudo-differenziale e quindi non-locale.

L'esistenza locale è abbastanza chiara. Più interessante è interrogarsi sull'esistenza di soluzioni globali nella variabile temporale, per dati iniziali piccoli. Questi risultati possono essere usati per provare la stabilità della soluzione nulla come soluzione dello stato stazionario.

La strategia consiste nel definire un opportuno spazio di funzioni e definire su tale spazio un operatore N tale che la soluzione che cerchiamo sia l'unico punto fisso di N. Per questa ragione usiamo il teorema di punto fisso di Banach. L'operatore N è definito in forma integrale mediante il principio di Duhamel. Per provare la contrattività di N va stimato il termine non-lineare. A tale scopo vengono in aiuto le disuguaglianze di tipo Gagliardo-Nirenberg, in particolare una forma generalizzata che proviene da [6], e alcune varianti da [8].

Questa tesi è organizzata nel seguente modo:

Nel primo capitolo abbiamo studiato l'equazione di Klein-Gordon con smorzamento classico. Abbiamo stimato l'energia e ottenuto uno decadimento esponenziale se supponiamo regolarità L^2 per i dati iniziali.

Nel capitolo 2 abbiamo studiato l'equazione delle onde con smorzamento visco-elastico. Abbiamo osservato che la sola regolarità L^2 dei dati non è sufficiente per ottenere decadimento. Allora abbiamo chiesto ulteriore regolarità ovvero $L^2 \cap L^1$. Sotto queste ipotesi abbiamo studiato il decadimento anche per derivate di ordine superiore della soluzione. Abbiamo ottenuto decadimento di tipo polinomiale.

Nel capitolo 3 abbiamo studiato l'equazione delle onde con smorzamento visco-elastico non-lineare con non-linearità di tipo $|u|^p$. I risultati circa questo modello sono già noti da [2]. Al fine di generalizzare questi risultati nei capitoli 4,5 e 6 abbiamo provato risultati di esistenza globale per alcuni esponenti ammissibili e per dati iniziali nello spazio $(H^2 \cap L^1) \times (L^2 \cap L^1)$.

Nel capitolo 4 abbiamo studiato l'equazione di Klein-Gordon con smorzamento e non-linearità di tipo $|u|^p$. Abbiamo ottenuto esistenza globale, nella variabile temporale, per tutti gli esponenti $p > 1$ e per dati iniziali piccoli, in virtù del decadimento esponenziale. Per questo modello è sufficiente supporre i dati iniziali in $H^1 \times L^2$.

Nel capitolo 5 ci siamo concentrati sull'equazione delle onde con smorzamento visco-elastico e non-linearità di tipo $|u_t|^p$. Abbiamo chiesto i dati iniziali in $(H^s \cap L^1) \times (H^{s-2} \cap L^1)$ con s grande. Sotto queste ipotesi abbiamo provato esistenza globale, nella variabile temporale, della soluzione per dati iniziali piccoli e per ogni esponente $p > s$.

Infine, nel capitolo 6 abbiamo studiato l'equazione delle onde con smorzamento visco-elastico e non-linearità del tipo $||D|^a u|^p$ con $a \in (0, 2)$. Inizialmente abbiamo chiesto regolarità $(H^2 \cap L^1) \times (L^2 \cap L^1)$ per i dati iniziali. Con questa regolarità abbiamo ottenuto esistenza globale, nella variabile temporale, ma con alcune restrizioni su a e sulla dimensione n. Allora abbiamo chiesto più regolarità dei dati iniziali, cioè $(H^s \cap L^1) \times (H^{s-2} \cap L^1)$ con s grande. Sotto queste ipotesi abbiamo trovato esistenza globale della soluzione per ogni esponente $p > s$.

Chapter 1

Wave models with mass and damping terms.

1.1 Linear theory

Let us consider the Cauchy problem for a wave model with mass and damping term

$$u_{tt} - \Delta u + m^2 u + b u_t = 0,\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x). \tag{1.1}$$

We assume that m^2 and b are two positive constants. In this chapter we want to prove well-posedness in Sobolev spaces and study the long time behaviour of suitable energies. We are also interested to understand what happens if m^2 is fixed and $b \to 0$, that is, we have the classical Klein-Gordon equation or if b is fixed and $m^2 \to 0$, that is, we have the classical damped wave model.

We introduce the function

$$w(t,x) = e^{\frac{b}{2}t} u(t,x).$$

If u satisfies the first equation, then w satisfies the partial differential equation of

Klein-Gordon type, but eventually with a negative mass term,

$$w_{tt} - \Delta w + \Big(m^2 - \frac{b^2}{4}\Big) w = 0, \ w(0,x) = \varphi(x), \ w_t(0,x) = \frac{b}{2}\varphi(x) + \psi(x).$$

Formal application of partial Fourier transformation gives the following ordinary differential equation for $v = v(t,\xi) = F_{x\to\xi}(w(t,x))(t,\xi)$ depending on parameter $|\xi|$:

$$v_{tt} + \Big(|\xi|^2 + \Big(m^2 - \frac{b^2}{4}\Big)\Big) v = 0, \ v(0,\xi) = v_0(\xi), \ v_t(0,\xi) = v_1(\xi),$$

where $v_0(\xi) := F(\varphi)(\xi)$ and $v_1(\xi) := \frac{b}{2}F(\varphi)(\xi) + F(\psi)(\xi)$.

We put

$$\alpha^2 := \Big| m^2 - \frac{b^2}{4} \Big|.$$

Now we can distinguish three cases.

<u>*First Case:*</u> $m^2 - \frac{b^2}{4} > 0$

We recognize the Klein- Gordon equation (positive mass term). We know its properties (Section 8.4 of [7]). The solution is given by

$$v(t,\xi) = \cos\Big(\sqrt{|\xi|^2 + \alpha^2}\ t\Big) v_0(\xi) + \frac{\sin\Big(\sqrt{|\xi|^2 + \alpha^2}\ t\Big)}{\sqrt{|\xi|^2 + \alpha^2}} v_1(\xi).$$

<u>*Second Case:*</u> $m^2 - \frac{b^2}{4} < 0$

We have a negative mass term. We know its property by the study of the classic damped wave equation (Section 12.2 of [7]).

On the set $\{\xi : |\xi| > \alpha\}$ we have

$$v(t,\xi) = \cos\Big(\sqrt{|\xi|^2 - \alpha^2}\ t\Big) v_0(\xi) + \frac{\sin\Big(\sqrt{|\xi|^2 - \alpha^2}\ t\Big)}{\sqrt{|\xi|^2 - \alpha^2}} v_1(\xi);$$

on the set $\{\xi : |\xi| < \alpha\}$ we have

$$\begin{aligned} v(t,\xi) &= \Big(\frac{v_0(\xi)}{2} - \frac{v_1(\xi)}{2\sqrt{\alpha^2 - |\xi|^2}}\Big) e^{-\sqrt{\alpha^2 - |\xi|^2} t} + \Big(\frac{v_0(\xi)}{2} + \frac{v_1(\xi)}{2\sqrt{\alpha^2 - |\xi|^2}}\Big) e^{\sqrt{\alpha^2 - |\xi|^2} t} \\ &= v_0(\xi)\cosh\Big(\sqrt{\alpha^2 - |\xi|^2} t\Big) + \frac{v_1(\xi)}{\sqrt{\alpha^2 - |\xi|^2}} \sinh\Big(\sqrt{\alpha^2 - |\xi|^2} t\Big). \end{aligned}$$

Third Case $m^2 - \frac{b^2}{4} = 0$

The equation becomes a classic wave equation (Section 11.3 of [7]). The solution is given by

$$v(t,\xi) = \cos\big(|\xi|t\big)v_0(\xi) + \frac{\sin\big(|\xi|t\big)}{|\xi|}v_1(\xi).$$

Taking into consideration that for regularity results only the behavior for large frequencies is of importance, we may conclude from the above representations the following result.

Theorem 1.1.1.

Let the data $\varphi \in H^s(\mathbb{R}^n)$ *and* $\psi \in H^{s-1}(\mathbb{R}^n)$, $s \in \mathbb{R}$, $n \geq 1$, *be given for the Cauchy problem*

$$u_{tt} - \Delta u + m^2 u + bu_t = 0,\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x).$$

Then for all $T > 0$ *there exists a unique distributional solution* $u \in C([0,T], H^s(\mathbb{R}^n)) \cap C^1([0,T], H^{s-1}(\mathbb{R}^n))$ *. If* $s \geq 0$*, then* u *is a Sobolev solution; if* $s \geq 1$*, then* u *is an energy solution; if* $s > \frac{n}{2} + 2$*, then* u *is even a classical solution.*

Remark 1.

This result is independent of the dimension n. It shows that there is no loss of regularity of solution in comparison with the data. Finally, we see, that mass or dissipation have no influence on the well-posedness in H^s spaces.

In the next step we introduce the total energy.

Definition 1.1.2.

We define the total energy

$$E_{KG}(u)(t) := \frac{1}{2}\int_{\mathbb{R}^n} \Big(|\nabla_x(u(t,x))|^2 + |u_t(t,x)|^2 + m^2|u(t,x)|^2\Big)\,dx,$$

where $u \in C([0,\infty), H^1(\mathbb{R}^n)) \cap C^1([0,\infty), L^2(\mathbb{R}^n))$.

Repeating the proof of Theorem 31 of [7], one can show the following result:

Theorem 1.1.3.
Let $u \in C([0,\infty), H^1(\mathbb{R}^n)) \cap C^1([0,\infty), L^2(\mathbb{R}^n))$ be a Sobolev solution of

$$u_{tt} - \Delta u + m^2 u + b u_t = 0,\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x),$$

where $\varphi \in H^1$ and $\psi \in L^2$. Then it holds

$$E_{KG}(u)(t) \le E_{KG}(u)(0) = \frac{1}{2}\big(\|\nabla\varphi\|_{L^2}^2 + \|\psi\|_{L^2} + m^2\|\psi\|_{L^2}^2\big) \quad \textit{for all } t \ge 0.$$

Remark 2.
The previous theorem shows that $E_{KG}(u)(t)$ is a decreasing function but it doesn't give us any information about eventual decay. We expect, in general, exponential type decay of $E_{KG}(u)(t)$. On the contrary we will observe a potential type decay of wave energy $E_{KG}(u)(t)$ if $m = 0$ and, finally, we have energy conservation of $E_{KG}(u)(t)$ for $b = 0$.

Theorem 1.1.4.
Let $u \in C([0,\infty), H^1(\mathbb{R}^n)) \cap C^1([0,\infty), L^2(\mathbb{R}^n))$ be the solution of the Cauchy problem

$$u_{tt} - \Delta u + m^2 u + b u_t = 0,\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x)$$

with data $\varphi \in H^1$ and $\psi \in L^2$. The coefficients m^2 and b are supposed to be positive. Then for any $t \ge 0$ it holds

$$\|u(t,\cdot)\|_{L^2} \lesssim \begin{cases} e^{-\frac{b}{2}t}\left(\|\varphi\|_{H^1} + \|\psi\|_{L^2}\right) & \textit{if } m^2 - \frac{b^2}{4} > 0; \\ e^{-\frac{b}{2}t}(1+t)\left(\|\varphi\|_{H^1} + \|\psi\|_{L^2}\right) & \textit{if } m^2 - \frac{b^2}{4} = 0; \\ e^{-\frac{m^2}{b}t}\left(\|\varphi\|_{H^1} + \|\psi\|_{L^2}\right) & \textit{if } m^2 - \frac{b^2}{4} < 0; \end{cases} \tag{1.2}$$

$$\|\nabla u(t,\cdot)\|_{L^2} \lesssim \begin{cases} e^{-\frac{b}{2}t}\left(\|\varphi\|_{H^1} + \|\psi\|_{L^2}\right) & \textit{if } m^2 - \frac{b^2}{4} > 0; \\ e^{-bt}\left(\|\varphi\|_{H^1}^2 + \|\psi\|_{L^2}^2\right) & \textit{if } m^2 - \frac{b^2}{4} = 0; \\ e^{-\frac{m^2}{b}t}(1+t)^{-1/2}\left(\|\varphi\|_{H^1} + \|\psi\|_{L^2}\right) & \textit{if } m^2 - \frac{b^2}{4} < 0; \end{cases} \tag{1.3}$$

$$\|u_t(t,\cdot)\|_{L^2} \lesssim \begin{cases} e^{-\frac{b}{2}t}\left(\|\varphi\|_{H^1} + \|\psi\|_{L^2}\right) & \textit{if } m^2 - \frac{b^2}{4} > 0; \\ e^{-\frac{b}{2}t}(1+t)\left(\|\varphi\|_{H^1} + \|\psi\|_{L^2}\right) & \textit{if } m^2 - \frac{b^2}{4} = 0; \\ e^{-\frac{m^2}{b}t}(1+t)^{-1}\left(\|\varphi\|_{H^1} + \|\psi\|_{L^2}\right) & \textit{if } m^2 - \frac{b^2}{4} < 0; \end{cases} \tag{1.4}$$

Consequently, the energy satisfies

$$E_{KG}(u)(t) \lesssim \begin{cases} e^{-bt}\big(\|\varphi\|_{H^1}^2 + \|\psi\|_{L^2}^2\big) & \textit{if } m^2 - \frac{b^2}{4} > 0; \\ e^{-\frac{b}{2}t}(1+t)\left(\|\varphi\|_{H^1} + \|\psi\|_{L^2}\right) & \textit{if } m^2 - \frac{b^2}{4} = 0; \\ \left(\dfrac{1}{1+t} + m^2\right) e^{-\frac{2m^2}{b}t}\big(\|\varphi\|_{H^1}^2 + \|\psi\|_{L^2}^2\big) & \textit{if } m^2 - \frac{b^2}{4} < 0. \end{cases} \tag{1.5}$$

Proof.

After using the dissipative transformation $w(t,x) = e^{\frac{b}{2}t}u(t,x)$, we know that w satisfies the partial differential equation

$$w_{tt} - \Delta w + \Big(m^2 - \frac{b^2}{4}\Big)w = 0,\ w(0,x) = \varphi(x),\ w_t(0,x) = \frac{b}{2}\varphi(x) + \psi(x).$$

Moreover, for any $a, b \in \mathbb{R}$ we have that $(a+b)^2 \leq 2(a^2+b^2)$. We will often use this inequality.

- $m^2 - \frac{b^2}{4} > 0$

  By using partial Fourier Transform and Plancherel's theorem we have that

$$\begin{aligned} \|u(t,\cdot)\|_{L^2}^2 = e^{-bt}\|w(t,\cdot)\|_{L^2}^2 &\lesssim e^{-bt}\int_{\mathbb{R}^n} \cos^2\Big(\sqrt{|\xi|^2+\alpha^2}\ t\Big)|v_0(\xi)|^2\,d\xi \\ &+ e^{-bt}\int_{\mathbb{R}^n} \frac{\sin^2\Big(\sqrt{|\xi|^2+\alpha^2}\ t\Big)}{|\xi|^2+\alpha^2}|v_1(\xi)|^2\,d\xi \\ &\lesssim e^{-bt}\int_{\mathbb{R}^n} |v_0(\xi)|^2 + |v_1(\xi)|^2\,d\xi \lesssim e^{-bt}\left(\|\varphi\|_{H^1}^2 + \|\psi\|_{L^2}^2\right). \end{aligned}$$

  In a very similar way one can prove that

$$\begin{aligned} \|\nabla u(t,\cdot)\|_{L^2}^2 &\lesssim e^{-bt}\left(\|\varphi\|_{H^1}^2 + \|\psi\|_{L^2}^2\right); \\ \|u_t(t,\cdot)\|_{L^2}^2 &\lesssim e^{-bt}\left(\|\varphi\|_{H^1}^2 + \|\psi\|_{L^2}^2\right). \end{aligned}$$

  More details can be found in the proof of the next case.

- $m^2 - \frac{b^2}{4} = 0$

  By using Fourier representation formula and remembering the fact that for a small α it holds $\dfrac{\sin\alpha}{\alpha} \le C$ we notice that

$$\begin{aligned}
\int_{\mathbb{R}^n} |w(t,x)|^2\,dx &\lesssim \int_{\mathbb{R}^n} \big|\cos\big(|\xi|t\big)v_0(\xi)\big|^2 + \left|\frac{\sin\big(|\xi|t\big)}{|\xi|}v_1(\xi)\right|^2 d\xi \\
&\lesssim \int_{\mathbb{R}^n} |v_0(\xi)|^2 d\xi + \int_{|\xi|\le\varepsilon} \left|\frac{\sin\big(|\xi|t\big)}{|\xi|t} t v_1(\xi)\right|^2 d\xi \\
&\quad + \int_{|\xi|\ge\varepsilon} \left|\frac{\sin\big(|\xi|t\big)}{|\xi|} v_1(\xi)\right|^2 d\xi \\
&\lesssim t^2\left(\|\varphi\|_{L^2}^2 + \|\psi\|_{L^2}^2\right).
\end{aligned}$$

By using the fact that for any $t \ge 0$, $t^2 e^{-bt} \lesssim (1+t)^2 e^{-bt}$, we have

$$\|u(t,\cdot)\|_{L^2}^2 \lesssim (1+t)^2 e^{-bt}\Big(\|\varphi\|_{H^1}^2 + \|\psi\|_{L^2}^2\Big).$$

Let us estimate the elastic energy. By using Fourier inversion formula we have that

$$\begin{aligned}
\|\nabla_x u(t,\cdot)\|_{L^2}^2 &= e^{-bt}\||\xi|v(t,\xi)\|_{L^2}^2 \\
&\lesssim e^{-bt}\left(\int_{\mathbb{R}^n} |\xi|^2\cos^2\left(|\xi|t\right)|v_0(\xi)|^2 d\xi + \int_{\mathbb{R}^n} \sin^2\left(|\xi|t\right)|v_1(\xi)|^2 d\xi\right) \\
&\lesssim e^{-bt}(\|\varphi\|_{H^1}^2 + \|\psi\|_{L^2}^2).
\end{aligned}$$

Finally let us estimate the kinetic energy. We have that

$$\hat{u}_t(t,\xi) = e^{-\frac{b}{2}t}\left[\left(-|\xi|\sin(|\xi|t) - \frac{b}{2}\cos(|\xi|t)\right)v_0(\xi) + \left(\cos(|\xi|t) - \frac{b}{2}\frac{\sin(|\xi|t)}{|\xi|}\right)v_1(\xi)\right].$$

Then in a very similar way to the previous case, we have that

$$\begin{aligned}
\|u_t(t,\cdot)\|_{L^2}^2 = \|\hat{u}_t(t,\cdot)\|_{L^2}^2 &\lesssim e^{-bt}\int_{\mathbb{R}^n} (|\xi|^2+1)|v_0(\xi)|^2 d\xi + e^{-bt}\int_{\mathbb{R}^n} (1+t^2)|v_1(\xi)|^2 d\xi \\
&\lesssim e^{-bt}(1+t)^2(\|\varphi\|_{H^1}^2 + \|\psi\|_{L^2}^2).
\end{aligned}$$

- $m^2 - \frac{b^2}{4} < 0$

  Let us estimate the potential energy $m^2\|u(t,\cdot)\|_{L^2}^2$.

Let $\hat{u}$ be the Fourier transform of u, so $\hat{u} = e^{-\frac{b}{2}t}v$. By Fourier inversion formula we know that $\|u(t,\cdot)\|_{L^2} = \|\hat{u}(t,\cdot)\|_{L^2} = e^{-\frac{b}{2}t}\|v(t,\cdot)\|_{L^2}$. We notice that $0 \leq \alpha = \sqrt{\dfrac{b^2}{4} - m^2} \leq \dfrac{b}{2}$ because $0 \leq m^2 \leq \frac{b^2}{4}$. We will distinguish several cases:

First Case $\{\xi : |\xi| > \alpha\}$

We notice that

$$\left|\frac{\sin\left(\sqrt{|\xi|^2 - \alpha^2}t\right)}{\sqrt{|\xi|^2 - \alpha^2}}\right| \lesssim \begin{cases} 1 & \text{if } |\xi| > 2\alpha, \\ t & \text{if } \alpha < |\xi| < 2\alpha. \end{cases}$$

By using Fourier representation formula we have that

$$\begin{aligned} \|\hat{u}(t,\cdot)\|^2_{L^2\{|\xi|>\alpha\}} &\leq e^{-bt}\Big(\int_{|\xi|>\alpha} |v_0(\xi)|^2\, d\xi + \int_{|\xi|>2\alpha} |v_1(\xi)|^2\, d\xi \\ &\quad + \int_{\alpha<|\xi|<2\alpha} \underbrace{\frac{\sin^2\left(\sqrt{|\xi|^2-\alpha^2}t\right)}{(|\xi|^2-\alpha^2)t^2}}_{\leq C}\, t^2\left|v_1(\xi)^2\right|\, d\xi\Big) \\ &\lesssim (1+t^2)e^{-bt}\int_{\mathbb{R}^n} |v_0(\xi)|^2 + |v_1(\xi)|^2\, d\xi. \end{aligned}$$

Second Case $\{\xi : |\xi| < \alpha\}$

We divide the interval $\big(0,\alpha\big)$ in two subintervals.

a) $\dfrac{\alpha}{2} \leq |\xi| < \alpha$

By using Fourier representation formula we get

$$|\hat{u}(t,\xi)| \leq e^{-\frac{b}{2}t}|v_0(\xi)|\underbrace{\left|\cosh\left(\sqrt{\alpha^2-|\xi|^2}t\right)\right|}_{\leq\cosh\left(\frac{\sqrt{3}\alpha}{2}t\right)} + e^{-\frac{b}{2}t}\underbrace{\left|\frac{\sinh\left(\sqrt{\alpha^2-|\xi|^2}t\right)}{\sqrt{\alpha^2-|\xi|^2}t}t\right|}_{\leq Ct\cosh\left(\frac{\sqrt{3}\alpha}{2}t\right)}|v_1(\xi)|.$$

But $\alpha \leq \dfrac{b}{2}$, so we notice that

$$\begin{aligned} te^{-\frac{b}{2}t}\cosh\Big(\frac{\sqrt{3}b}{4}t\Big) &= te^{-\frac{b}{2}t}\,\frac{e^{\frac{\sqrt{3}b}{4}t} + e^{-\frac{\sqrt{3}b}{4}t}}{2} = te^{-\frac{b}{2}t+\frac{\sqrt{3}b}{4}t}\,\frac{1+e^{-\frac{\sqrt{3}b}{2}t}}{2} \\ &\leq te^{\frac{\sqrt{3}-2}{4}bt} \lesssim e^{-\frac{\delta}{2}bt} \end{aligned}$$

with $\delta \in (0, \frac{2-\sqrt{3}}{2})$. It follows

$$\int_{\frac{\alpha}{2}<|\xi|<\alpha} |\hat{u}(t,\xi)|^2\, d\xi \leq Ce^{-\delta bt}\int_{\mathbb{R}^n} |v_0(\xi)|^2 + |v_1(\xi)|^2\, d\xi.$$

b) $0 < |\xi| < \dfrac{\alpha}{2}$

We remember that for $0 < q < 1$ it holds

$$\sqrt{1-q^2} = 1 - \frac{q^2}{2} + o(q^2),$$

so by using the fact that $0 \le \alpha \le \dfrac{b}{2}$ we can conclude that

$$-\frac{b}{2} + \sqrt{\alpha^2 - |\xi|^2} \lesssim -\frac{b}{2} + \alpha\left(1 - \frac{|\xi|^2}{2\alpha^2}\right) \lesssim -\frac{m^2}{b} - \frac{|\xi|^2}{2\alpha}. \tag{1.6}$$

As in the proof of Theorem 34 of [7] we notice that

$$\int_{|\xi|<\frac{\alpha}{2}} |\hat{u}(t,\xi)|^2\, d\xi \lesssim \int_{|\xi|<\frac{\alpha}{2}} \left(|v_0(\xi)|^2 + |v_1(\xi)|^2\right) \big(\underbrace{e^{-bt+\sqrt{\alpha^2-|\xi|^2}t}}_{\lesssim e^{\left(-\frac{2m^2}{b}-\frac{|\xi|^2}{\alpha}\right)t}} + \underbrace{e^{-bt-\sqrt{\alpha^2-|\xi|^2}t}}_{\le e^{-bt}}\big)\, d\xi$$

$$\lesssim e^{-\frac{2m^2}{b}t} \int_{\mathbb{R}^n} \left(e^{-\frac{|\xi|^2}{\alpha}t} + 1\right) \left(|v_0(\xi)|^2 + |v_1(\xi)|^2\right)\, d\xi.$$

Summarizing we proved that

$$m^2 \|u(t,\cdot)\|_{L^2}^2 \lesssim m^2 e^{-\frac{2m^2}{b}t} (\|\varphi\|_{H^1}^2 + \|\psi\|_{L^2}^2).$$

Let us estimate the elastic energy. By using Fourier inversion formula we have that

$$\|\nabla_x u(t,\cdot)\|_{L^2} = e^{-\frac{b}{2}t} \||\xi| v(t,\xi)\|_{L^2}.$$

In a very similar way to the previous case and by using a similar approach for proving Theorem 34 of [7] we can show that

$$\int_{|\xi|>\alpha} |\xi|^2 |\hat{u}(t,\xi)|^2\, d\xi \lesssim t^2 e^{-bt} \int_{\mathbb{R}^n} |\xi|^2 |v_0(\xi)|^2 + |v_1(\xi)|^2\, d\xi;$$
$$\int_{\frac{\alpha}{2}<|\xi|<\alpha} |\xi|^2 |\hat{u}(t,\xi)|^2\, d\xi \lesssim e^{-\delta bt} \int_{\mathbb{R}^n} |\xi|^2 |v_0(\xi)|^2 + |v_1(\xi)|^2\, d\xi \tag{1.7}$$

with $\delta \in (0, \frac{2-\sqrt{3}}{2})$. Let us focus on small frequencies. If $|\xi| \le \frac{\alpha}{2}$, then by using the inequality (1.6) as in the proof of Theorem 34 of [7]

$$\int_{|\xi|<\frac{\alpha}{2}} |\xi|^2 |\hat{u}(t,\xi)|^2\, d\xi \lesssim e^{-bt} \int_{|\xi|<\frac{\alpha}{2}} \left(|\xi|^2 |v_0(\xi)|^2 + |\xi|^2 \frac{|v_1(\xi)|^2}{\alpha^2}\right) d\xi$$
$$+ e^{\frac{-2m^2}{b}t} \int_{|\xi|<\frac{\alpha}{2}} \left(|v_0(\xi)|^2 + \frac{|v_1(\xi)|^2}{\alpha^2}\right) |\xi|^2 e^{-\frac{|\xi|^2 t}{2\alpha}}\, d\xi.$$

By using the norm inequality $\|\cdot\|_{L^2} \leq \|\cdot\|_{L^\infty}\|\cdot\|_{L^2}$ for large t we get for the second integral of the right-hand side of the last inequality

$$e^{-\frac{2m^2}{b}t}\int_{|\xi|<\frac{\alpha}{2}}\Big(|v_0(\xi)|^2+\frac{|v_1(\xi)|^2}{\alpha^2}\Big)|\xi|^2 e^{-\frac{|\xi|^2 t}{\alpha}}\,d\xi$$
$$\lesssim e^{-\frac{2m^2}{b}t}\frac{\alpha}{t}\sup_{|\xi|<\frac{\alpha}{2},t\geq 1}\frac{|\xi|^2 t}{\alpha}e^{-\frac{|\xi|^2 t}{\alpha}}\int_{\mathbb{R}^n}\Big(|v_0(\xi)|^2+\frac{|v_1(\xi)|^2}{\alpha^2}\Big)\,d\xi$$
$$\lesssim e^{-\frac{2m^2}{b}t}\frac{\alpha}{t}\int_{\mathbb{R}^n}\Big(|v_0(\xi)|^2+\frac{|v_1(\xi)|^2}{\alpha^2}\Big)\,d\xi.$$

Summarizing we have shown for small frequencies and for large t

$$\int_{|\xi|<\frac{\alpha}{2}}|\xi|^2|\hat{u}(t,\xi)|^2\,d\xi \lesssim e^{-\frac{2m^2}{b}t}\frac{\alpha}{1+t}\int_{\mathbb{R}^n}|\xi|^2|v_0(\xi)|^2+|v_0(\xi)|^2+|v_1(\xi)|^2 d\xi.$$

Of course for small t and small frequencies it holds

$$e^{-\frac{|\xi|^2 t}{2\alpha}} \lesssim \frac{1}{1+t}.$$

By using the fact that $2m \leq b$ we have that $e^{-bt} \lesssim e^{-\frac{2m^2}{b}t}$ and so we can conclude

$$\|\nabla u(t,\cdot)\|^2_{L^2} \lesssim \frac{e^{-\frac{2m^2}{b}t}}{1+t}(\|\varphi\|^2_{H^1}+\|\psi\|^2_{L^2}).$$

Let us estimate the kinetic energy. By using Fourier inversion formula we have that

$$\|u_t(t,\cdot)\|_{L^2} = \|\hat{u}_t(t,\cdot)\|_{L^2}.$$

On the set $\{|\xi| > \alpha\}$ we have that

$$\hat{u}_t(t,\xi) = e^{-\frac{b}{2}t}\Bigg[-\frac{b}{2}\cos\Big(\sqrt{|\xi|^2-\alpha^2}\;t\Big)v_0(\xi) - \frac{b}{2}\frac{\sin\Big(\sqrt{|\xi|^2-\alpha^2}\;t\Big)}{\sqrt{|\xi|^2-\alpha^2}}v_1(\xi)$$
$$+\cos\Big(\sqrt{|\xi|^2-\alpha^2}\;t\Big)v_1(\xi) - \sqrt{|\xi|^2-\alpha^2}\sin\Big(\sqrt{|\xi|^2-\alpha^2}\;t\Big)v_0(\xi)\Bigg].$$

On the set $\{|\xi| < \alpha\}$ we have that

$$\hat{u}_t(t,\xi) = e^{-\frac{b}{2}t}\Bigg[-\frac{b}{2}\cosh\Big(\sqrt{\alpha^2-|\xi|^2}\;t\Big)v_0(\xi) - \frac{b}{2}\frac{\sinh\Big(\sqrt{\alpha^2-|\xi|^2}\;t\Big)}{\sqrt{\alpha^2-|\xi|^2}}v_1(\xi)$$
$$+\cosh\Big(\sqrt{\alpha^2-|\xi|^2}\;t\Big)v_1(\xi) + \sqrt{\alpha^2-|\xi|^2}\sinh\Big(\sqrt{\alpha^2-|\xi|^2}\;t\Big)v_0(\xi)\Bigg].$$

By using the inequality

$$\sqrt{\alpha^2 - |\xi|^2}\sinh\left(\sqrt{\alpha^2 - |\xi|^2}t\right) \lesssim (|\xi|^2+1)\,\frac{\sinh\left(\sqrt{\alpha^2 - |\xi|^2}t\right)}{\sqrt{\alpha^2 - |\xi|^2}},$$

each term can be estimated with a similar approach of the elastic energy. We can conclude

$$\|u_t(t,\cdot)\|_{L^2}^2 \lesssim \frac{e^{-\frac{2m^2}{b}t}}{(1+t)^2}(\|\varphi\|_{H^1}^2 + \|\psi\|_{L^2}^2).$$

Summarizing we completed our proof. □

Remark 3.
As we explained at the beginning we are interested to understand what happens if

- m^2 is fixed and $b \to 0$;
- b is fixed and $m^2 \to 0$.

In the first case we may assume that $m^2 - \frac{b^2}{4} > 0$, so we may use the first line of (1.5). We expect that the solution has a similar behaviour to the solution of the classical Klein-Gordon equation, i.e., a result about conservation of energy. In fact, if $b \to 0$, then the dissipative transformation is meaningless. We notice that $e^{-\delta t} \to 1$, so we lose the decay.

Let us consider the second case. We may assume that $m^2 - \frac{b^2}{4} < 0$ so we may use the third line of (1.5). We know that

$$E(u)(t) \lesssim \left(\frac{1}{1+t} + m^2\right) e^{-\frac{2m^2}{b}t}(\|\varphi\|_{H^1}^2 + \|\psi\|_{L^2}^2).$$

If $m^2 \to 0$, then we see that the energy decay is of potential type and not of exponential type.

Chapter 2

Wave models with visco-elastic damping.

2.1 Linear theory

Let us consider the Cauchy problem for a wave model with structural (visco-elastic) damping

$$u_{tt} - \Delta u - \Delta u_t = 0,\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x). \tag{2.1}$$

The principal part of the partial differential operator in the last equation is Δu_t, but we can interpret this term as a visco-elastic damping term to the wave equation. Let us calculate the characteristic equation of this operator. We have that:

$$-\tau^2 + |\xi|^2 - i|\xi|^2\tau - = 0.$$

Calculating the characteristic roots, we can consider this model as a *"hyperbolic like"* model. Formal application of partial Fourier transformation gives the following ordinary differential equation for $v = v(t,\xi) = F_{x\to\xi}(u(t,x))(t,\xi)$ depending on parameter $|\xi|$:

$$v_{tt} + |\xi|^2 v_t + |\xi|^2 v = 0,\ v(0,\xi) = v_0(\xi),\ v_t(0,\xi) = v_1(\xi),$$

where $v_0(\xi) := F(\varphi)(\xi)$ and $v_1(\xi) := F(\psi)(\xi)$. The solution is given on the set $\{\xi : |\xi| > 2\}$ by

$$\begin{aligned} v(t,\xi) &= e^{-\frac{|\xi|^2 t}{2}} \Bigg[\frac{v_0(\xi)\Big(|\xi|\sqrt{|\xi|^2-4}+|\xi|^2\Big)+2v_1(\xi)}{2|\xi|\sqrt{|\xi|^2-4}} e^{\frac{|\xi|\sqrt{|\xi|^2-4}\,t}{2}} \\ &\qquad + \frac{v_0(\xi)\Big(|\xi|\sqrt{|\xi|^2-4}-|\xi|^2\Big)-2v_1(\xi)}{2|\xi|\sqrt{|\xi|^2-4}} e^{\frac{-|\xi|\sqrt{|\xi|^2-4}\,t}{2}} \Bigg] \\ &= e^{-\frac{|\xi|^2 t}{2}} \Bigg[\frac{|\xi|^2 v_0(\xi)+2v_1(\xi)}{|\xi|\sqrt{|\xi|^2-4}} \sinh\Bigg(\frac{|\xi|\sqrt{|\xi|^2-4}}{2}t\Bigg) \\ &\qquad + v_0(\xi)\cosh\Bigg(\frac{|\xi|\sqrt{|\xi|^2-4}}{2}t\Bigg)\Bigg] \end{aligned}$$

and on the set $\{\xi : |\xi| < 2\}$ by

$$v(t,\xi) = e^{-\frac{|\xi|^2 t}{2}} \Bigg[\frac{|\xi|^2 v_0(\xi)+2v_1(\xi)}{|\xi|\sqrt{4-|\xi|^2}} \sin\Bigg(\frac{|\xi|\sqrt{4-|\xi|^2}}{2}t\Bigg) + v_0(\xi)\cos\Bigg(\frac{|\xi|\sqrt{4-|\xi|^2}}{2}t\Bigg)\Bigg].$$

In a very similar way of the proof to Theorem 32 of [7] we can prove the following well-posedness theorem (here we choose data $(\varphi,\psi) \in H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n)$ as in the classical wave case):

Theorem 2.1.1.
Let the data $\varphi \in H^s(\mathbb{R}^n)$ and $\psi \in H^{s-1}(\mathbb{R}^n)$, $s \in \mathbb{R}$, $n \geq 1$ be given for the Cauchy problem

$$u_{tt} - \Delta u - \Delta u_t = 0,\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x).$$

Then there exists a uniquely determined distributional solution $u \in C([0,T], H^s(\mathbb{R}^n)) \cap C^1([0,T], H^{s-1}(\mathbb{R}^n))$. If $s \geq 0$, then u is a Sobolev solution; if $s \geq 1$, then u is an energy solution; if $s > \frac{n}{2}+3$, then u is a classical solution.

Remark 4. If we consider Theorem 1.1.1 we notice that in that case we have a classical solution for $s > \frac{n}{2}+1$ and now we need $s > \frac{n}{2}+1$ to obtain a classical solution. This difference comes out because in the first case the principal part of the operator is Δu but now the principal part of the operator is Δu_t.

Now let us introduce the total energy.

Definition 2.1.2.

We define the total wave type energy

$$E_W(u)(t) := \frac{1}{2}\int_{\mathbb{R}^n}\Big(|\nabla_x(u(t,x))|^2 + |u_t(t,x)|^2\Big)\,dx$$

for all $u \in C([0,\infty), H^1(\mathbb{R}^n)) \cap C^1([0,\infty), L^2(\mathbb{R}^n))$.

Repeating the proof of Theorem 31 of [7] one can show the following result:

Theorem 2.1.3.

Let $u \in C([0,\infty), H^2(\mathbb{R}^n)) \cap C^1([0,\infty), H^1(\mathbb{R}^n))$ be a Sobolev solution of

$$u_{tt} - \Delta u - \Delta u_t = 0,\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x),$$

where $\varphi \in H^2$ and $\psi \in H^1$. Then it holds

$$E_W(u)(t) \le E_W(u)(0) = \frac{1}{2}\big(\|\nabla\varphi\|^2_{L^2} + \|\psi\|_{L^2}\big) \quad \textit{for all } t \ge 0.$$

Let u be the solution of the Cauchy problem

$$u_{tt} - \Delta u - \Delta u_t = 0,\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x)$$

with data $\varphi \in H^1$ and $\psi \in L^2$. In the following we are interested to apply phase space analysis (similar to Section 1) to estimate the energy under the before proposed regularity of data.

Estimate of the elastic energy

We will distinguish several cases.

Case 1 $\{\xi : |\xi| > 2\}$

We divide the interval $\big(2, +\infty\big)$ in two subintervals.

a) $4 \le |\xi| < +\infty$

By using the fact that $\sqrt{q^2-1} \leq q - \dfrac{1}{2q}$ for $q > 1$ we can estimate as follows:

$$e^{-\frac{|\xi|^2 t}{2}} \cosh\left(\frac{|\xi|\sqrt{|\xi|^2-4}\,t}{2}\right) = e^{-\frac{|\xi|^2 t}{2}} \frac{e^{\frac{|\xi|\sqrt{|\xi|^2-4}\,t}{2}} + e^{-\frac{|\xi|\sqrt{|\xi|^2-4}\,t}{2}}}{2}$$

$$= e^{-\frac{|\xi|^2}{2}t + \frac{|\xi|\sqrt{|\xi|^2-4}}{2}t} \cdot \frac{1 + e^{-|\xi|\sqrt{|\xi|^2-4}\,t}}{2} \lesssim e^{-\frac{|\xi|^2}{2}t + \frac{|\xi|^2}{2}t - t} \lesssim e^{-t};$$

and, obviously, it holds

$$e^{-\frac{|\xi|^2 t}{2}} \sinh\left(\frac{|\xi|\sqrt{|\xi|^2-4}\,t}{2}\right) \lesssim e^{-t}.$$

So, we have

$$\||\xi|v(t,\xi)\|^2_{L^2\{\xi>4\}} = \int_{|\xi|>4} |\xi|^2|v(t,\xi)|^2 d\xi \lesssim e^{-2t} \int_{|\xi|>4} |\xi|^2|v_0(\xi)|^2 d\xi$$

$$+ e^{-2t} \int_{|\xi|>4} \frac{4|\xi|^2}{|\xi|^4 - 4|\xi|^2}|v_1(\xi)|^2 + \frac{|\xi|^4}{|\xi|^4 - 4|\xi|^2}|\xi|^2|v_0(\xi)|^2 d\xi$$

$$\lesssim e^{-2t} \int_{\mathbb{R}^n} |v_1(\xi)|^2 + |\xi|^2|v_0(\xi)|^2 d\xi.$$

b) $2 < |\xi| < 4$

By Weierestrass theorem, by using the continuity of function $\frac{\sinh(x)}{x}$, we may estimate as follows:

$$\||\xi|v(t,\xi)\|^2_{L^2\{2<|\xi|<4\}} = \int_{|\xi|>4} |\xi|^2|v(t,\xi)|^2 d\xi \lesssim e^{-2t} \int_{2<|\xi|<4} |\xi|^2 v_0(\xi)^2 d\xi$$

$$+ t^2 e^{-2t} \int_{2<|\xi|<4} (4|v_1(\xi)|^2 + |\xi|^2|v_0(\xi)|^2) \underbrace{\frac{\sinh^2\left(\frac{|\xi|\sqrt{|\xi|^2-4}}{2}t\right)}{\frac{|\xi|^2(|\xi|^2-4)t^2}{4}}}_{\frac{\sinh^2\alpha}{\alpha^2} \leq C} \frac{|\xi|^2}{4} d\xi$$

$$\lesssim t^2 e^{-2t} \int_{\mathbb{R}^n} |v_1(\xi)|^2 + |v_0(\xi)|^2 d\xi.$$

Case 2 $\{\xi : |\xi| < 2\}$

We divide the interval $(0,2)$ in two subintervals.

a) $1 \leq |\xi| < 2$

We have

$$\||\xi| v(t,\xi)\|^2_{L^2\{1<\xi<2\}} = \int_{1<|\xi|<2} |\xi|^2 |v(t,\xi)|^2 d\xi \lesssim e^{-t} \int_{1<|\xi|<2} |\xi|^2 |v_0(\xi)|^2$$

$$+ Ct^2 e^{-t} \int_{1<|\xi|<2} (4|v_1(\xi)|^2 + |\xi|^2 |v_0(\xi)|^2) \underbrace{\frac{\sin^2\left(\frac{|\xi|\sqrt{4-|\xi|^2}}{2} t\right)}{\frac{|\xi|^2(4-|\xi|^2)t^2}{4}}}_{\frac{\sin^2\alpha}{\alpha^2} \leq C} \frac{|\xi|^2}{4} d\xi$$

$$\lesssim t^2 e^{-t} \int_{\mathbb{R}^n} |v_1(\xi)|^2 + |v_0(\xi)|^2 d\xi.$$

b) $|\xi| < 1$

On this subinterval we notice that

$$e^{\frac{|\xi|^2 t}{2}} |\xi||v(t,\xi)| \leq |\xi||v_0(\xi)| \left[\frac{|\xi|}{\sqrt{4-|\xi|^2}} \sin\left(\frac{|\xi|\sqrt{4-|\xi|^2}}{2} t\right) + \cos\left(\frac{|\xi|\sqrt{4-|\xi|^2}}{2} t\right) \right]$$

$$+ |v_1(\xi)| \frac{2}{\sqrt{4-|\xi|^2}} \sin\left(\frac{|\xi|\sqrt{4-|\xi|^2}}{2} t\right)$$

$$\lesssim \Big(|\xi||v_0(\xi)| + |v_1(\xi)| \Big).$$

So, for large t we have

$$\||\xi| v(t,\xi)\|^2_{L^2\{|\xi|<1\}} \lesssim \int_{|\xi|<1} e^{-|\xi|^2 t} |\xi|^2 |v_0(\xi)|^2 d\xi + \int_{|\xi|<1} e^{-|\xi|^2 t} |v_1(\xi)|^2 \, d\xi$$

$$\lesssim \frac{1}{t} \sup_{|\xi|<1, t>1} \left(t|\xi|^2 e^{-|\xi|^2 t} \right) \int_{|\xi|<1} |v_0(\xi)|^2 d\xi + \int_{|\xi|<1} |v_1(\xi)|^2 d\xi$$

$$\lesssim \frac{1}{1+t} \|v_0\|^2_{L^2} + \|v_1\|^2_{L^2}.$$

Of course for small t and for small frequencies we simply use

$$e^{-|\xi|^2 t} \lesssim \frac{1}{1+t}.$$

Summarizing we have shown that

$$\|\nabla u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{1}{2}} \|\varphi\|_{H^1} + \|\psi\|_{L^2}.$$

Estimate of kinetic energy

As in the estimate of elastic energy we can transfer the energy into the phase space, indeed $\|u_t(t,\cdot)\|^2_{L^2} = \|v_t(t,\cdot)\|^2_{L^2}$.

Case 1 $\{\xi : |\xi| > 2\}$

We have

$$v_t(t,\xi) = e^{-\frac{|\xi|^2 t}{2}}\left[v_1(\xi)\cosh\left(\frac{|\xi|\sqrt{|\xi|^2-4}}{2}t\right) - |\xi|\frac{v_1(\xi)+2v_0(\xi)}{\sqrt{|\xi|^2-4}}\sinh\left(\frac{|\xi|\sqrt{|\xi|^2-4}}{2}t\right)\right].$$

Repeating the reasoning to estimate the elastic energy gives

$$\|u_t(t,\cdot)\|^2_{L^2\{\xi>2\}} \lesssim (1+t^2)e^{-2t}\int_{\mathbb{R}^n}|v_1(\xi)|^2 + |v_0(\xi)|^2 d\xi.$$

Case 2 $\{\xi : |\xi| < 2\}$

We have

$$v_t(t,\xi) = e^{-\frac{|\xi|^2 t}{2}}\left[v_1(\xi)\cos\left(\frac{|\xi|\sqrt{4-|\xi|^2}}{2}t\right) - |\xi|\frac{v_1(\xi)+2v_0(\xi)}{\sqrt{4-|\xi|^2}}\sin\left(\frac{|\xi|\sqrt{4-|\xi|^2}}{2}t\right)\right].$$

We divide the interval $(0,2)$ in two subintervals.

a) $1 \le |\xi| < 2$

Repeating the reasoning to estimate the elastic energy we get

$$\begin{aligned}\|u_t(t,\cdot)\|^2_{L^2\{1<|\xi|<2\}} &\lesssim e^{-t}\int_{1<|\xi|<2}|v_1(\xi)|^2 d\xi \\ &\quad + t^2 e^{-t}\int_{1<|\xi|<2}(4|v_0(\xi)|^2 + |v_1(\xi)|^2)\frac{|\xi|^4}{4}d\xi \\ &\lesssim t^2 e^{-t}\int_{\mathbb{R}^n}|v_1(\xi)|^2 + |v_0(\xi)|^2 d\xi.\end{aligned}$$

b) $|\xi| < 1$

On this subinterval we notice that

$$\begin{aligned}e^{\frac{|\xi|^2 t}{2}}|v_t(t,\xi)| \le |v_1(\xi)|&\left[\cos\left(\frac{|\xi|\sqrt{4-|\xi|^2}}{2}t\right) + \frac{|\xi|}{\sqrt{4-|\xi|^2}}\sin\left(\frac{|\xi|\sqrt{4-|\xi|^2}}{2}t\right)\right] \\ &+ |v_0(\xi)|\frac{2|\xi|}{\sqrt{4-|\xi|^2}}\sin\left(\frac{|\xi|\sqrt{4-|\xi|^2}}{2}t\right) \lesssim |\xi||v_0(\xi)| + |v_1(\xi)|.\end{aligned}$$

Repeating the reasoning to estimate the elastic energy gives

$$\|v_t(t,\xi)\|^2_{L^2\{|\xi|<1\}} \lesssim (1+t)^{-1}\|\varphi\|^2_{L^2} + \|\psi\|^2_{L^1}.$$

Summarizing we proved that

$$\|u_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{1}{2}}\|\varphi\|_{H^1} + \|\psi\|_{L^2}.$$

We should notice that with the supposed regularity of initial data $\varphi \in H^1$ and $\psi \in L^2$ we lose the decay of the wave energy. Let us try to understand what happens if we assume additional regularity of the data. Let us suppose additional L^1 regularity for the data, that is,

$$\varphi \in H^1 \cap L^1, \text{ and } \psi \in L^2 \cap L^1$$

Estimating the small frequencies part we see that

$$\begin{aligned}\||\xi|v(t,\xi)\|^2_{L^2\{|\xi|<1\}} &\lesssim \int_{|\xi|<1} e^{-|\xi|^2t}|\xi|^2|v_0(\xi)|^2 d\xi + C\int_{|\xi|<1} e^{-|\xi|^2t}|v_1(\xi)|^2 \\ &\lesssim \|v_0(\xi)^2\|_{L^\infty\{|\xi<1|\}}\int_{|\xi|<1}|\xi|^2 e^{-|\xi|^2t}d\xi + \|v_1(\xi)^2\|_{L^\infty\{|\xi<1|\}}\int_{|\xi|<1} e^{-|\xi|^2t}d\xi.\end{aligned}$$

Let us estimate the integrals on the right-hand sides of the last inequality. By using polar coordinates for large t, for $a \geq 0$ we have

$$\begin{aligned}\int_{|\xi|<1}|\xi|^a e^{-|\xi|^2t}d\xi &= C\int_0^1 e^{-r^2t}r^{a+n-1}dr = C\frac{1}{\sqrt{t^{a+n-1}}}\int_0^1 e^{-r^2t}(r\sqrt{t})^{n-1}dr \\ &\leq Ct^{-\frac{n+a}{2}}\int_0^{\sqrt{t}} e^{-s^2}s^{n-1}ds \lesssim (1+t)^{-\frac{n+a}{2}}.\end{aligned} \tag{2.2}$$

For small t we can also obtain the previous inequality, indeed for $|\xi| < 1$ it holds

$$e^{-|\xi|^2t} \lesssim \frac{1}{(1+t)^{(n+a)/2}}.$$

By using the fact that $\|v_0(\xi)^2\|_{L^\infty\{|\xi<1|\}} \leq \|v_0\|^2_{L^\infty(\mathbb{R}^n)}$ by Riemann - Lebesgue Theorem we have

$$\||\xi|v(t,\xi)\|^2_{L^2\{|\xi|<1\}} \lesssim (1+t)^{-\frac{n+2}{2}}\|\varphi\|^2_{L^1} + (1+t)^{-\frac{n}{2}}\|\psi\|^2_{L^1}.$$

Combining this with the strong decay estimate in the zone $|\xi| > 1$, under the above assumption for the regularity of data we may conclude

$$\|\nabla u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n+2}{4}}\|\varphi\|_{H^1\cap L^1} + (1+t)^{-\frac{n}{4}}\|\psi\|_{L^2\cap L^1}.$$

In a very similar way one can show that with the supposed regularity of data $(\varphi,\psi) \in (H^1 \cap L^1) \times (L^2 \cap L^1)$ it holds

$$\|u_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n+2}{4}}\|\varphi\|_{H^1\cap L^1} + (1+t)^{-\frac{n}{4}}\|\psi\|_{L^2\cap L^1}.$$

This means, that the wave energy decays. Summarizing we have derived the two estimates

$$\|(\nabla u, u_t)(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{1}{2}}\|\varphi\|_{H^1} + \|\psi\|_{L^2},$$
$$\|(\nabla u, u_t)(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n+2}{4}}\|\varphi\|_{H^1\cap L^1} + (1+t)^{-\frac{n}{4}}\|\psi\|_{L^2\cap L^1}.$$

Interpolating both inequalities we may expect for $m \in (1,2)$ the estimate

$$\|(\nabla u, u_t)(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n(2-m)}{4m}-\frac{1}{2}}\|\varphi\|_{H^1\cap L^m} + (1+t)^{-\frac{n(2-m)}{4m}}\|\psi\|_{L^2\cap L^m}.$$

Now we prove such estimates. Let m' be the conjugate exponent to m. Estimating the zone of small frequencies by using Hölder's inequality and Hausdorff-Young inequality, we see that

$$\begin{aligned}
\||\xi|v(t,\xi)\|^2{}_{L^2\{|\xi|<1\}} &\lesssim \int_{|\xi|<1} e^{-|\xi|^2 t}\big(|\xi|^2|v_0(\xi)|^2 + |v_1(\xi)|^2\big)d\xi \\
&\lesssim \|v_0\|^2_{L^{m'}}\left(\int_{|\xi|<1}\big(|\xi|^2 e^{-|\xi|^2 t}\big)^{\frac{m}{2-m}}d\xi\right)^{\frac{2-m}{m}} + \|v_1\|^2_{L^{m'}}\left(\int_{|\xi|<1}\big(e^{-|\xi|^2 t}\big)^{\frac{m}{2-m}}d\xi\right)^{\frac{2-m}{m}} \\
&\lesssim \|\varphi\|^2_{L^m}\left(\int_{|\xi|<1}\big(|\xi|^2 e^{-|\xi|^2 t}\big)^{\frac{m}{2-m}}d\xi\right)^{\frac{2-m}{m}} + \|\psi\|^2_{L^m}\left(\int_{|\xi|<1} e^{-|\xi|^2\frac{tm}{2-m}}d\xi\right)^{\frac{2-m}{m}}.
\end{aligned}$$

Let us only estimate the last integrals on the right-hand side. By using polar coordinates for large t we have

$$\begin{aligned}
\int_{|\xi|<1} e^{-|\xi|^2\frac{tm}{2-m}}d\xi &= C\int_0^{\frac{tm}{2-m}} e^{-r^2\frac{tm}{2-m}}r^{n-1}dr = C\left(\frac{2-m}{mt}\right)^{n/2}\int_0^{\frac{tm}{2-m}} se^{-s^2}ds \\
&\lesssim \left(\frac{1+mt}{2-m}\right)^{-n/2}.
\end{aligned}$$

In a very similar way to the case $m=1$ this inequality holds even for small t. Hence, it holds

$$\||\xi|v(t,\xi)\|^2_{L^2\{|\xi|<1\}} \lesssim \left(\frac{1+mt}{2-m}\right)^{-\frac{n}{2}\cdot\frac{2-m}{m}-1}\|\varphi\|^2_{L^m} + \left(\frac{1+mt}{2-m}\right)^{-\frac{n}{2}\cdot\frac{2-m}{m}}\|\psi\|^2_{L^m}.$$

Combining with a stronger result for large frequencies, we can conclude that with the supposed regularity of data it holds

$$\|\nabla u(t,\cdot)\|_{L^2} \lesssim \left(\frac{1+mt}{2-m}\right)^{-\frac{n}{4}\cdot\frac{2-m}{m}-\frac{1}{2}}\|\varphi\|_{H^1\cap L^m} + \left(\frac{1+mt}{2-m}\right)^{-\frac{n}{4}\cdot\frac{2-m}{m}}\|\psi\|_{L^2\cap L^m}.$$

In a very similar way one get the desired estimate for the kinetic energy.

Due to its utility in the semi-linear model we want to estimate the L^2 norm of the solution. By Plancherel theorem we know that $\|u(t,\cdot)\|_{L^2} = \|v(t,\cdot)\|_{L^2}$. After the division of phase space into several zones a very similar way to the previous cases we get

$$\|v(t,\xi)\|^2_{L^2\{|\xi|>2\}} \lesssim t^2e^{-t}\int_{\mathbb{R}^n} |v_1(\xi)|^2 + |v_0(\xi)|^2\, d\xi;$$
$$\|v(t,\xi)\|^2_{L^2\{1<|\xi|<2\}} \lesssim t^2e^{-t}\int_{\mathbb{R}^n} |v_1(\xi)|^2 + |v_0(\xi)|^2\, d\xi.$$

Let us focus to small frequencies. If we suppose $|\xi| < 1$, then we have

$$\begin{aligned}\|v(t,\xi)\|^2_{L^2\{|\xi|<1\}} \lesssim & \int_{|\xi|<1} e^{-|\xi|^2t}\left[\frac{|\xi|^4|v_0(\xi)|^2 + |v_1(\xi)|^2}{|\xi|^2(4-|\xi|^2)}\sin^2\left(\frac{|\xi|\sqrt{4-|\xi|^2}}{2}t\right)\right] d\xi \\ & + \int_{|\xi|<1} e^{-|\xi|^2t}|v_0(\xi)|^2\cos^2\left(\frac{|\xi|\sqrt{4-|\xi|^2}}{2}t\right) d\xi.\end{aligned}$$

If we assume only $H^1 \times L^2$ regularity for the data as in the previous case we lose decay. In fact, we can only estimate

$$\begin{aligned}\|v(t,\xi)\|^2_{L^2\{|\xi|<1\}} &\lesssim \int_{|\xi|<1} e^{-|\xi|^2t}\left(|\xi|^2+1\right)|v_0(\xi)|^2\, d\xi + \int_{|\xi|<1} t^2e^{-|\xi|^2t}|v_1(\xi)|^2 d\xi \\ &\lesssim \|\varphi\|^2_{L^2} + t^2\|\psi\|^2_{L^2}.\end{aligned}$$

So, summarizing we showed that

$$\|u(t,\cdot)\|_{L^2} \lesssim \|\varphi\|_{H^1} + t\|\psi\|_{L^2}.$$

Let us ask for corresponding estimates under additional regularity for the data, for example, $(H^1 \cap L^1) \times (L^2 \cap L^1)$. Let us distinguish two cases:

- if $n \geq 3$, by using polar coordinates we obtain

$$\begin{aligned}\|v(t,\xi)\|^2_{L^2\{|\xi|<1\}} &\lesssim \|v_0\|^2_\infty \int_0^1 e^{-r^2t}\frac{r^2+4-r^2}{(4-r^2)}r^{n-1}\, dr \\ &\quad + \|v_1\|^2_\infty \int_0^1 e^{-r^2t}\frac{1}{r^2(4-r^2)}r^{n-1}dr \\ &\lesssim \|v_0\|^2_\infty \int_0^1 e^{-r^2t}r^{n-1}dr + \|v_1\|^2_\infty \int_0^1 e^{-r^2t}r^{n-3}dr.\end{aligned}$$

  Proceeding as (2.2), one has

$$\int_0^1 e^{-r^2t}r^{n-3+2\alpha}dr \lesssim (1+t)^{-\frac{n-2+2\alpha}{2}}\,. \tag{2.3}$$

Taking $\alpha = 0, 1$, it follows that

$$\|v(t,\xi)\|^2_{L^2\{|\xi|<1\}} \lesssim (1+t)^{-\frac{n}{2}}\|v_0\|^2_\infty + (1+t)^{-\frac{n-2}{2}}\|v_1\|^2_\infty.$$

- If $n = 2$, by using polar coordinates we obtain

$$\|v(t,\xi)\|^2_{L^2\{|\xi|<1\}} \lesssim \|v_0\|_\infty \int_0^1 e^{-r^2 t} r\, dr + \|v_1\|^2_\infty \int_0^1 \frac{e^{-r^2 t}}{r^2(4-r^2)} \sin^2\left(\frac{r\sqrt{4-r^2}}{2} t\right) r\, dr.$$

The first integral on the right-hand can be controlled by using (2.3) Let us focus on the second one. For any $t > 1$, we put

$$\mathrm{I} = \int_0^{\frac{1}{t}} \frac{e^{-r^2 t}}{r^2(4-r^2)} \sin^2\left(\frac{r\sqrt{4-r^2}}{2} t\right) r\, dr,$$

$$\mathrm{II} = \int_{\frac{1}{t}}^1 \frac{e^{-r^2 t}}{r^2(4-r^2)} \sin^2\left(\frac{r\sqrt{4-r^2}}{2} t\right) r\, dr.$$

Now let us separately deal with these two terms. In the first term we notice that $rt < 1$, so

$$\left|\sin\left(\frac{r\sqrt{4-r^2}}{2} t\right)\right| \lesssim r\sqrt{4-r^2}\, t \lesssim rt; \tag{2.4}$$

then

$$\mathrm{I} \lesssim \int_0^{\frac{1}{t}} \frac{(tr)^2}{r} \lesssim 1.$$

Now let us estimate the second term. We notice

$$\mathrm{II} \lesssim \int_{\frac{1}{t}}^1 \frac{e^{-r^2 t}}{r}\, dr = \int_{\frac{\sqrt{t}}{t}}^{\sqrt{t}} \frac{e^{-s^2}}{s}\, ds = \left[\log(s) e^{-s^2}\right]_{\frac{\sqrt{t}}{t}}^{\sqrt{t}} + \int_{\frac{\sqrt{t}}{t}}^{\sqrt{t}} 2s e^{-s^2} \log(s)\, ds.$$

Let us evaluate the integral in the right hand side. We have that:

$$\begin{aligned}\int_{\frac{\sqrt{t}}{t}}^{\sqrt{t}} 2s e^{-s^2} \log(s)\, ds &= \int_{\frac{\sqrt{t}}{t}}^{1} 2s e^{-s^2} \log(s)\, ds + \int_1^{\sqrt{t}} 2s e^{-s^2} \log(s)\, ds \\ &\lesssim \int_1^{+\infty} 2s e^{-s^2} \log(s)\, ds = C.\end{aligned}$$

And so

$$\mathrm{II} \lesssim \log(t) e^{-t} - \log\left(\frac{1}{t}\right) e^{-\frac{1}{t}} + 1 \lesssim \log(e+t).$$

So, summarizing the estimates for $t > 1$ we get

$$\int_0^1 \frac{e^{-r^2 t}}{r^2(4-r^2)} \sin^2\left(\frac{r\sqrt{4-r^2}}{2} t\right) r\, dr \lesssim \log(e+t).$$

If we now suppose $t \le 1$ then $rt \le 1$. By using (2.4) we have that

$$\int_0^1 e^{-r^2 t} \frac{e^{-r^2 t}}{r^2(4-r^2)} \sin^2\left(\frac{r\sqrt{4-r^2}}{2} t\right) r\,dr \lesssim \int_0^1 \frac{rt}{r}\,dr \lesssim 1 \lesssim \log(e+t).$$

Therefore by using Riemann-Lebesgue theorem we proved that

$$\|u(t,\cdot)\|_{L^2} \lesssim \begin{cases} (1+t)^{-\frac{n}{4}} \|\varphi\|_{H^1\cap L^1} + (1+t)^{-\frac{n-2}{4}} \|\psi\|_{L^2\cap L^1} & \text{if } n \ge 3; \\ (1+t)^{-\frac{1}{2}} \|\varphi\|_{H^1\cap L^1} + \log(e+t)\|\psi\|_{L^2\cap L^1} & \text{if } n = 2. \end{cases}$$

Finally, we suppose $\varphi \in H^1 \cap L^m$ and $\psi \in L^2 \cap L^m$ with $m \in (1,2)$. By using Hölder's inequality in a very similar way to the previous case one can prove that

$$\|u(t,\cdot)\|_{L^2} \lesssim \begin{cases} \left(\frac{1+mt}{2-m}\right)^{-\frac{n}{4}\cdot\frac{2-m}{m}} \|\varphi\|_{H^1\cap L^m} + \left(\frac{1+mt}{2-m}\right)^{-\frac{nm-4m+2}{4(2-m)}} \|\psi\|_{L^2\cap L^m} & \text{if } n \ge 3; \\ \left(\frac{1+mt}{2-m}\right)^{-\frac{n}{4}\cdot\frac{2-m}{m}} \|\varphi\|_{H^1\cap L^m} + \left(\frac{1+mt}{2-m}\right)^{\frac{1-m}{4(2-m)}} \log(e+t)\|\psi\|_{L^2\cap L^m} & \text{if } n = 2; \end{cases}$$

We can resume all the results in the following theorem:

Theorem 2.1.4.
Let u be the solution of the Cauchy problem

$$u_{tt} - \Delta u - \Delta u_t = 0,\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x)$$

with data $\varphi \in H^1 \cap L^m$ and $\psi \in L^2 \cap L^m$, where $m \in [1,2)$. Then for any $t \ge 0$ the following estimates are valid:

$$\|u(t,\cdot)\|_{L^2} \lesssim \begin{cases} \left(\frac{1+mt}{2-m}\right)^{-\frac{n}{4}\cdot\frac{2-m}{m}} \|\varphi\|_{H^1\cap L^m} + \left(\frac{1+mt}{2-m}\right)^{-\frac{nm-4m+2}{4(2-m)}} \|\psi\|_{L^2\cap L^m} & \text{if } n \ge 3; \\ \left(\frac{1+mt}{2-m}\right)^{-\frac{n}{4}\cdot\frac{2-m}{m}} \|\varphi\|_{H^1\cap L^m} + \left(\frac{1+mt}{2-m}\right)^{\frac{1-m}{4(2-m)}} \log(e+t)\|\psi\|_{L^2\cap L^m} & \text{if } n = 2; \end{cases} \tag{2.5}$$

$$\|\nabla u(t,\cdot)\|_{L^2} \lesssim \left(\frac{1+mt}{2-m}\right)^{-\frac{n}{4}\cdot\frac{2-m}{m}-\frac{1}{2}} \|\varphi\|_{H^1\cap L^m} + \left(\frac{1+mt}{2-m}\right)^{-\frac{n}{4}\cdot\frac{2-m}{m}} \|\psi\|_{L^2\cap L^m}; \tag{2.6}$$

$$\|u_t(t,\cdot)\|_{L^2} \lesssim \left(\frac{1+mt}{2-m}\right)^{-\frac{n}{4}\cdot\frac{2-m}{m}-\frac{1}{2}} \|\varphi\|_{H^1\cap L^m} + \left(\frac{1+mt}{2-m}\right)^{-\frac{n}{4}\cdot\frac{2-m}{m}} \|\psi\|_{L^2\cap L^m}. \tag{2.7}$$

Therefore, the wave type energy satisfies

$$E_W(u)(t) \lesssim \left(\frac{1+mt}{2-m}\right)^{-\frac{n}{2}\cdot\frac{2-m}{m}} \left(\|\varphi\|^2_{H^1\cap L^m} + \|\psi\|^2_{L^2\cap L^m}\right). \tag{2.8}$$

If $m = 2$, then for any $t \geq 0$ the following estimates hold:

$$\|u(t,\cdot)\|_{L^2} \lesssim \|\varphi\|_{H^1} + (1+t)\|\psi\|_{L^2}; \tag{2.9}$$

$$\|\nabla u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{1}{2}}\|\varphi\|_{H^1} + \|\psi\|_{L^2}; \tag{2.10}$$

$$\|u_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{1}{2}}\|\varphi\|_{H^1} + \|\psi\|_{L^2}. \tag{2.11}$$

Consequently, the wave type energy satisfies

$$E_W(u)(t) \lesssim \big(\|\varphi\|^2_{H^1} + \|\psi\|^2_{L^2}\big). \tag{2.12}$$

We can generalize the previous result to energies of higher order.

Theorem 2.1.5.
Let u be the solution of the Cauchy problem

$$u_{tt} - \Delta u - \Delta u_t = 0,\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x)$$

with data $\varphi \in H^{\beta+1} \cap L^m$ and $\psi \in H^\beta \cap L^m$, where $\beta > 0$ and $m \in [1,2)$. Then for any $t \geq 0$ the following estimates are satisfied:

$$\||D|^{\beta+1}u(t,\cdot)\|_{L^2} \lesssim \Big(\frac{1+mt}{2-m}\Big)^{-\frac{n+2\beta}{4}\frac{m}{2-m}-\frac{1}{2}}\|\varphi\|_{H^{\beta+1}\cap L^m} + \Big(\frac{1+mt}{2-m}\Big)^{-\frac{n+2\beta}{4}\frac{m}{2-m}}\|\psi\|_{H^\beta\cap L^m}; \tag{2.13}$$

$$\||D|^{\beta}u_t(t,\cdot)\|_{L^2} \lesssim \Big(\frac{1+mt}{2-m}\Big)^{-\frac{n+2\beta}{4}\frac{m}{2-m}-\frac{1}{2}}\|\varphi\|_{H^{\beta+1}\cap L^m} + \Big(\frac{1+mt}{2-m}\Big)^{-\frac{n+2\beta}{4}\frac{m}{2-m}}\|\psi\|_{H^\beta\cap L^m}. \tag{2.14}$$

If $m = 2$, then for any $t \geq 0$ the following estimates are satisfied:

$$\||D|^{\beta+1}u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{\beta+1}{2}}\|\varphi\|_{H^{\beta+1}} + (1+t)^{-\frac{\beta}{2}}\|\psi\|_{H^\beta}; \tag{2.15}$$

$$\||D|^{\beta}u_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{\beta+1}{2}}\|\varphi\|_{H^{\beta+1}} + (1+t)^{-\frac{\beta}{2}}\|\psi\|_{H^\beta}. \tag{2.16}$$

Proof.
Let us first consider the case $m = 2$. As in the previous cases we transfer the energy into the phase space, so we have

$$\||D|^{\beta+1}u(t,\cdot)\|_{L^2} = \||\xi|^{\beta+1}v(t,\cdot)\|_{L^2}, \quad \||D|^{\beta}u_t(t,\cdot)\|_{L^2} = \||\xi|^{\beta}v_t(t,\cdot)\|_{L^2}.$$

Estimates of elastic energies of higher order

Case 1 $\{|\xi| > 2\}$

By using the same arguments to derive estimates for the classical elastic energy one can show that

$$\||\xi|^{\beta+1}v(t,\xi)\|_{L^2\{|\xi|>2\}} \lesssim t^2e^{-t}\int_{\mathbb{R}^n}|\xi|^{2\beta}|v_1(\xi)|^2 + |\xi|^{2\beta+2}|v_0(\xi)|^2d\xi.$$

Case 2 $\{\xi : |\xi| < 2\}$

We divide the interval $\big[0,2)$ in two subintervals.

a) $1 \le |\xi| < 2$

We have

$$\begin{aligned}\||\xi|^{\beta+1}v(t,\xi)\|^2_{L^2\{1<\xi<2\}} &= \int_{1<|\xi|<2}|\xi|^{2\beta+2}|v(t,\xi)|^2d\xi \lesssim e^{-t}\int_{1<|\xi|<2}|\xi|^{2\beta+2}|v_0(\xi)|^2 \\ &+ t^2e^{-t}\int_{1<|\xi|<2}(4|\xi|^{2\beta}|v_1(\xi)|^2 + |\xi|^{2\beta+2}|v_0(\xi)|^2)\underbrace{\frac{\sin^2\left(\frac{|\xi|\sqrt{4-|\xi|^2}}{2}t\right)}{\frac{|\xi|^2(4-|\xi|^2)t^2}{4}}}_{\frac{\sin^2\alpha}{\alpha^2}\le C}d\xi \\ &\lesssim t^2e^{-t}\int_{\mathbb{R}^n}|\xi|^{2\beta}|v_1(\xi)|^2 + |\xi|^{2\beta+2}|v_0(\xi)|^2d\xi.\end{aligned}$$

b) $|\xi| < 1$

As for the zero order energy we have for large t

$$\begin{aligned}\||\xi|^{\beta+1}v(t,\xi)\|^2_{L^2\{|\xi|<1\}} &\lesssim \int_{|\xi|<1}e^{-|\xi|^2t}|\xi|^{2\beta+2}|v_0(\xi)|^2d\xi + \int_{|\xi|<1}e^{-|\xi|^2t}|\xi|^{2\beta}|v_1(\xi)|^2\,d\xi \\ &\lesssim \frac{1}{t^{\beta+1}}\sup_{|\xi|<1,t>1}\left(t|\xi|^2)^{\beta+1}\right)e^{-|\xi|^2t}\int_{|\xi|<1}|v_0(\xi)|^2d\xi \\ &+ \frac{1}{t^{\beta+1}}\sup_{|\xi|<1,t>1}\left(t|\xi|^2)^{\beta+1}\right)e^{-|\xi|^2t}\int_{|\xi|<1}|v_1(\xi)|^2d\xi \\ &\lesssim \frac{1}{(1+t)^{\beta+1}}\|\varphi\|^2_{H^{\beta+1}} + \frac{1}{(1+t)^{\beta}}\|\psi\|^2_{H^\beta}.\end{aligned}$$

As in the previous it is evident that this inequality holds even for small t. Summarizing we have shown that for any $t \ge 0$ it holds

$$\||D|^{\beta+1}u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{\beta+1}{2}}\|\varphi\|_{H^{\beta+1}} + (1+t)^{-\frac{\beta}{2}}\|\psi\|_{H^\beta}.$$

In a very similar way one can see that

$$\||D|^{\beta}u_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{\beta+1}{2}}\|\varphi\|_{H^{\beta+1}} + (1+t)^{-\frac{\beta}{2}}\|\psi\|_{H^\beta}.$$

Now let us consider $m \in [1,2)$. By using the same approach of the proof of inequality (2.6) and (2.7) one can prove (2.13) and (2.14) and this completes our proof. $\square$

Remark 5. Previous theorem says that wave equation with visco-elastic damping shows a parabolic effect holds; higher order energies have a faster decay.

Until now we suppose that the initial data $(\varphi, \psi) \in (H^{s+1} \times H^s)$ with eventually additional regularity. We will see that this kind of regularity is not enough for semi-linear models. So we will suppose $(\varphi, \psi) \in (H^{s+2} \times H^s)$ with eventually additional regularity. In such a case the energy estimates of Theorem 2.1.5 will be modified as in the following:

Theorem 2.1.6.

Let u be the solution of the Cauchy problem

$$u_{tt} - \Delta u - \Delta u_t = 0,\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x)$$

with data $\varphi \in H^{\beta+2} \cap L^m$ and $\psi \in H^\beta \cap L^m$, where $\beta \geq 0$ and $m \in [1,2)$. Then for any $t \geq 0$ the following estimates are satisfied:

$$\||D|^{\beta+2}u(t,\cdot)\|_{L^2} \lesssim \left(\frac{1+mt}{2-m}\right)^{-\frac{n+2(\beta+1)}{4}\frac{m}{2-m}-\frac{1}{2}} \|\varphi\|_{H^{\beta+2}\cap L^m} + \left(\frac{1+mt}{2-m}\right)^{-\frac{n+2(\beta+1)}{4}\frac{m}{2-m}} \|\psi\|_{H^\beta \cap L^m}; \tag{2.17}$$

$$\||D|^{\beta}u_t(t,\cdot)\|_{L^2} \lesssim \left(\frac{1+mt}{2-m}\right)^{-\frac{n+2\beta}{4}\frac{m}{2-m}-\frac{1}{2}} \|\varphi\|_{H^{\beta+2}\cap L^m} + \left(\frac{1+mt}{2-m}\right)^{-\frac{n+2\beta}{4}\frac{m}{2-m}} \|\psi\|_{H^\beta \cap L^m}. \tag{2.18}$$

If $m = 2$, then for any $t \geq 0$ the following estimates are satisfied:

$$\||D|^{\beta+2}u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{\beta+2}{2}} \|\varphi\|_{H^{\beta+2}} + (1+t)^{-\frac{\beta+1}{2}} \|\psi\|_{H^\beta}; \tag{2.19}$$

$$\||D|^{\beta}u_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{\beta+2}{2}} \|\varphi\|_{H^{\beta+2}} + (1+t)^{-\frac{\beta+1}{2}} \|\psi\|_{H^\beta}. \tag{2.20}$$

In this way the parabolic effect holds, that is, higher order energies have a faster decay.

Proof.

We will proceed as before. We know that

$$\||D|^{\beta+2}u(t,\cdot)\|_{L^2} = \||\xi|^{\beta+2}v(t,\cdot)\|_{L^2};$$
$$\||D|^{\beta}u_t(t,\cdot)\|_{L^2} = \||\xi|^{\beta}v_t(t,\cdot)\|_{L^2}.$$

Let us estimate the term $|\xi|^{\beta+2}v(t,\xi)$. By using representation formula of the solution, with a similar reasoning to the Theorem 2.1.5, we can estimate as follow:

- if $|\xi| > 2$

$$\begin{aligned}||\xi|^{\beta+2}v(t,\xi)| &\lesssim e^{-\frac{t}{2}}\left[|\xi|^{\beta+2}\left(\frac{|\xi|^2}{|\xi|\sqrt{|\xi|^2-4}}+1\right)|v_0(\xi)| + |\xi|^{\beta+2}\frac{2}{|\xi|\sqrt{|\xi|^2-4}}|v_1(\xi)|\right]\\ &\lesssim e^{-\frac{t}{2}}\left[|\xi|^{\beta+2}|v_0(\xi)| + |\xi|^{\beta}|v_1(\xi)|\right];\end{aligned}$$

- if $|\xi| < 2$

$$\begin{aligned}||\xi|^{\beta+2}v(t,\xi)| &\lesssim e^{-\frac{|\xi|^2 t}{2}}\left[|\xi|^{\beta+2}\left(\frac{|\xi|^2}{|\xi|\sqrt{|\xi|^2-4}}+1\right)|v_0(\xi)| + |\xi|^{\beta+2}\frac{2}{|\xi|\sqrt{|\xi|^2-4}}|v_1(\xi)|\right]\\ &\lesssim e^{-\frac{|\xi|^2 t}{2}}\left[|\xi|^{\beta+2}|v_0(\xi)| + |\xi|^{\beta+1}|v_1(\xi)|\right].\end{aligned}$$

We can do the same reasoning to estimate $|\xi|^\beta v_t(t,\xi)$, it means

- if $|\xi| > 2$

$$\begin{aligned}||\xi|^{\beta}v_t(t,\xi)| &\lesssim e^{-\frac{t}{2}}\left[|\xi|^{\beta}\frac{2|\xi|}{\sqrt{|\xi|^2-4}}|v_0(\xi)| + |\xi|^{\beta}\left(\frac{|\xi|}{\sqrt{|\xi|^2-4}}+1\right)|v_1(\xi)|\right]\\ &\lesssim e^{-\frac{t}{2}}\left[|\xi|^{\beta+2}|v_0(\xi)| + |\xi|^{\beta}|v_1(\xi)|\right];\end{aligned}$$

- if $|\xi| < 2$

$$\begin{aligned}||\xi|^{\beta}v_t(t,\xi)| &\lesssim e^{-\frac{|\xi|^2 t}{2}}\left[|\xi|^{\beta}\frac{2|\xi|}{\sqrt{|\xi|^2-4}}|v_0(\xi)| + |\xi|^{\beta}\left(\frac{|\xi|}{\sqrt{|\xi|^2-4}}+1\right)|v_1(\xi)|\right]\\ &\lesssim e^{-\frac{|\xi|^2 t}{2}}\left[|\xi|^{\beta+1}|v_0(\xi)| + |\xi|^{\beta}|v_1(\xi)|\right].\end{aligned}$$

With the same strategy of the proof of Theorems 2.1.4 and 2.1.5 we can conclude this proof.

□

Remark 6. Let us consider initial data $(\varphi,\psi) \in H^{s+2} \times H^s$ and u the solution to (2.1) with initial condition (φ,ψ). We consider $\{\varphi_k\} \in C_0^\infty$ and $\{\psi_k\} \in C_0^\infty$ such that $\varphi_k \to \varphi$ in H^{s+2} and $\psi_k \to \psi$ in H^s. If u_k is the solution to (2.1) with initial condition (φ_k,ψ_k), by Theorem 2.1.1 we have that u_k is a classical solution and . By using inequalities of Theorem 2.1.6 I have that $u_k \to u$, so $u \in C([0,T], H^{s+2}(\mathbb{R}^n)) \cap C^1([0,T], H^s(\mathbb{R}^n))$.

Chapter 3

Semi-linear visco-elastic damped wave models with source term.

3.1 Main tools

Now we study the Cauchy problem for the semi-linear visco-elastic damped wave model with source term. This problem has already been treated in [2]. We will show some methods and some approach for the study of semi-linear problem and we expect to use this methods to study some other semi-linear problems in next chapters.

We consider the Cauchy problem

$$u_{tt} - \Delta u - \Delta u_t = |u|^p,\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x). \tag{3.1}$$

We will assume some regularity for the data. So we give the following definition:

Definition 3.1.1.
Let $m \in [1,2]$ and $k \geq 0$. We define:

$$\mathcal{D}_m^k := \left(H^{k,m} \cap L^1\right) \times \left(L^m \cap L^1\right),$$

with the norm

$$\|(v_0, v_1)\|_{\mathcal{D}_m^k} := \|v_0\|_{L^1} + \|v_0\|_{H^{k,m}} + \|v_1\|_{L^1} + \|v_1\|_{L^m}.$$

Our aim is to prove global (in time) existence results.

Our main tools are Duhamel's principle, Gagliardo-Nirenberg inequality and Banach's fixed-point theorem. Since we are dealing with semi-linear visco-elastic damped waves with constant coefficients in the linear part the application of Duhamel's principle leads to the following: Let us consider the linear problem

$$(u_{lin})_{tt} - \Delta(u_{lin}) - \Delta(u_{lin})_t = 0,\ (u_{lin})(0,x) = \varphi(x),\ (u_{lin})_t(0,x) = \psi(x).$$

According with Chapter 2 one has

$$(u_{lin})(t,x) = G_0(t,x) *_x \varphi(x) + G_1(t,x) *_x \psi(x);$$

being

$$F_{x\to\xi}G_0(t,\xi) = \begin{cases} e^{-\frac{|\xi|^2 t}{2}} \left[\frac{|\xi|}{\sqrt{4-|\xi|^2}} \sinh\left(\frac{|\xi|\sqrt{4-|\xi|^2}}{2} t \right) + \cosh\left(\frac{|\xi|\sqrt{4-|\xi|^2}}{2} t \right) \right] & \text{if } |\xi| > 2; \\ e^{-\frac{|\xi|^2 t}{2}} \left[\frac{|\xi|}{\sqrt{4-|\xi|^2}} \sin\left(\frac{|\xi|\sqrt{4-|\xi|^2}}{2} t \right) + \cos\left(\frac{|\xi|\sqrt{4-|\xi|^2}}{2} t \right) \right] & \text{if } |\xi| < 2. \end{cases} \tag{3.2}$$

and

$$F_{x\to\xi}G_1(t,\xi) = \begin{cases} \frac{2e^{-\frac{|\xi|^2 t}{2}}}{|\xi|\sqrt{4-|\xi|^2}} \sinh\left(|\xi| \frac{|\xi|\sqrt{4-|\xi|^2}}{2} t \right) & \text{if } |\xi| > 2 \\ \frac{2e^{-\frac{|\xi|^2 t}{2}}}{|\xi|\sqrt{4-|\xi|^2}} \sin\left(|\xi| \frac{|\xi|\sqrt{4-|\xi|^2}}{2} t \right) & \text{if } |\xi| < 2 \end{cases} \tag{3.3}$$

Using Duhamel's principle any solution to (3.1) satisfies

$$u(t,x) = G_0(x,t) *_{(x)} \varphi(x) + G_1(x,t) *_{(x)} \psi(x) + \int_0^t G_1(t-s,x) *_{(x)} |u(s,x)|^p ds. \tag{3.4}$$

Having in mind Theorems 2.1.4 and 2.1.6, with $m = 1$, we define for all $t \geq 0$

$$X(t) := C([0,t], H^2) \cap C^1([0,t], L^2),$$

with the norm

$$\|w\|_{X(t)} := \sup_{0\leq\tau\leq t} \left(\sum_{k=0}^{2} (1+\tau)^{\frac{n-2+2k}{4}} \|\nabla^k w\|_{L^2} + (1+\tau)^{\frac{n}{4}} \|w_t\|_{L^2} \right). \tag{3.5}$$

We notice that we are changing regularity of data with respect to Theorem 2.1.5 and 2.1.6. More precisely, if $n = 2$, then we have

$$\|w\|_{X(t)} := \sup_{0\le\tau\le t}\left((\log(e+t))^{-1}\|w\|_{L^2} + \sum_{k=1}^{2}(1+\tau)^{\frac{n-2+2k}{4}}\|\nabla^k w\|_{L^2} + (1+\tau)^{\frac{n}{4}}\|w_t\|_{L^2}\right)$$

but the term $(\log(e+t))^{-1}$ brings no additional difficulties so we will ignore it. It is easy to prove that $\left(X(t), \|\cdot\|_{X(t)}\right)$ is a Banach space. We also define

$$\|w\|_{X_0(t)} := \sup_{0\le\tau\le t}\left(\sum_{k=0}^{2}(1+\tau)^{\frac{n-2+2k}{4}}\|\nabla^k w\|_{L^2}\right). \tag{3.6}$$

Our aim is to prove that for any data (φ, ψ) in a certain space A the operator N which is defined for any $w \in X(t)$ by

$$N:\ w \to Nw := G_0(x,t) *_{(x)} \varphi(x) + G_1(x,t) *_{(x)} \psi(x) + \int_0^t G_1(t-s,x) *_{(x)} |w(s,x)|^p ds$$

satisfies the following estimates:

$$\|Nu\|_{X(t)} \lesssim \|(\varphi,\psi)\|_A + \|u\|^p_{X_0(t)}, \tag{3.7}$$

$$\|Nu - Nv\|_{X(t)} \lesssim \|u-v\|_{X_0(t)}\left(\|u\|^{p-1}_{X_0(t)} + \|v\|^{p-1}_{X_0(t)}\right) \tag{3.8}$$

uniformly with respect to $t \ge 0$. By (3.7) it follows that for small initial data N maps $X(t)$ into itself and by (3.8) we can deduce that N has a unique fixed point u which is also the unique solution of (3.1). One can see [2] for more details.

Since all of the constants are independent of t we can take $t \to +\infty$ and we gain a local and a global existence result simultaneously.

Finally, we see that the definition of $X(t)$ is chosen in an appropriate way to obtain the decay estimates for the solution to the semi-linear problem, too.

During the proof a special role shall play different applications of Gagliardo-Nirenberg inequality to control suitable L^2-norms of non-linear terms. In particular, we will use the following:

Theorem 3.1.2 (Gagliardo-Nirenberg inequality).
Let $u \in H^k$. Then

$$\|u(s,\cdot)\|_{L^q} \lesssim \|u(s,\cdot)\|^{(1-\theta_k(q))}_{L^2}\|\nabla^k u(s,\cdot)\|^{\theta_k(q)}_{L^2}, \tag{3.9}$$

where $k = 1, 2$ *and*

$$\theta_k(q) := \frac{n}{k}\left(\frac{1}{2} - \frac{1}{q}\right), \qquad 2 \le q \le \frac{2n}{n-2k}.$$

For the proof one can see [5].

3.2 Global existence and decay behaviour

Theorem 3.2.1.
Let $A := (H^2 \cap L^1) \times (L^2 \cap L^1)$*,* $n \ge 2$ *and let*

$$p \in [2, n/(n-4)] \text{ be such that } p > 1 + \frac{3}{n-1}. \tag{3.10}$$

Then there exists $\varepsilon > 0$ *such that for any* $(\varphi, \psi) \in A$ *with* $\|(\varphi, \psi)\|_A < \varepsilon$ *there exists a unique solution* $u \in C([0, \infty), H^2) \cap C^1([0, \infty), L^2)$ *to* (3.1).

Moreover the solution, its first derivative in time and its derivatives in space up to the second order satisfy the decay estimates satisfies the following estimates:

$$\|u(t,\cdot)\|_{L^2} \lesssim \begin{cases} (1+t)^{-\frac{n-2}{4}}\|(\varphi,\psi)\|_A & \text{if } n \ge 3; \\ \log(e+t)\|(\varphi,\psi)\|_A & \text{if } n = 2; \end{cases}$$

$$\|\nabla u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{4}}\|(\varphi,\psi)\|_A;$$

$$\|\nabla^2 u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n+2}{4}}\|(\varphi,\psi)\|_A;$$

$$\|u_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{4}}\|(\varphi,\psi)\|_A.$$

Proof.
We give an idea of this proof. For the complete proof one can see [2].

As we said before we just need to prove (3.7) and (3.8). Let us prove (3.7). By using (2.5) we have that

$$\|Nu(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n-2}{4}}\|(\varphi,\psi)\|_{\mathcal{D}_2^0} + \int_0^t (1+t-s)^{-\frac{n-2}{4}}\||u(s,\cdot)|^p\|_{L^2\cap L^1} ds. \tag{3.11}$$

Let us estimate the derivative of Nu. We will distinguish the cases $s \in [0, t/2]$ and $s \in [t/2, t]$ to control the integral in (3.11). In particular we will use the $(L^2 \cap L^1) - L^2$ estimates (2.6), (2.7) and (2.17) if $s \in [0, t/2]$ and the $L^2 - L^2$ estimates (2.10), (2.11) and (2.19) if $s \in [t/2, t]$. So we have for $(j, |\alpha|) \in \{(1,0); (0,1); (0,2)\}$ the estimates

$$\begin{aligned} \|\partial_t^j \partial_x^\alpha N u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n+2(j+|\alpha|-1)}{4}} \|(\varphi,\psi)\|_{\mathcal{D}_2^{|\alpha|}} \\ + \int_0^{t/2} (1+t-s)^{-\frac{n+2(j+|\alpha|-1)}{4}} \||u(s,\cdot)|^p\|_{L^2\cap L^1} ds \\ + \int_{t/2}^t (1+t-s)^{-\frac{j-|\alpha|-1}{2}} \||u(s,\cdot)|^p\|_{L^2} ds. \end{aligned} \tag{3.12}$$

We estimate

$$\begin{aligned} \||u(s,\cdot)|^p\|_{L^2\cap L^1} &\lesssim \|u(s,\cdot)\|_{L^p}^p + \|u(s,\cdot)\|_{L^{2p}}^p, \\ \||u(s,\cdot)|^p\|_{L^2} &\lesssim \|u(s,\cdot)\|_{L^{2p}}^p. \end{aligned}$$

Then by using (3.9) for $q = p$, $q = 2p$ and $k = 2$ we obtain

$$\||u(s,\cdot)|^p\|_{L^2\cap L^1} \lesssim \|u\|_{X_0(s)}^p (1+s)^{-\frac{p(n-1)-n}{2}}, \tag{3.13}$$

$$\||u(s,\cdot)|^p\|_{L^2} \lesssim \|u\|_{X_0(s)}^p (1+s)^{-\frac{p(n-1)-n/2}{2}}. \tag{3.14}$$

We notice that to obtain the previous two estimates, it is essential (3.10).

Taking account of

$$(1+t-s) \approx (1+t) \ \textit{if}\ s \in [0, t/2]; \quad (1+s) \approx (1+t) \ \textit{if}\ s \in [t/2, t] \tag{3.15}$$

after using (3.13) and (3.15) we may estimate (3.11) as follows:

$$\begin{aligned} \|Nu(t,\cdot)\|_{L^2} \lesssim & (1+t)^{-\frac{n-2}{4}} \|(\varphi,\psi)\|_{\mathcal{D}_2^0} \\ & + \|u\|_{X_0(t)}^p (1+t)^{-\frac{n-2}{4}} \int_0^{t/2} (1+s)^{-\frac{p(n-1)-n}{2}} ds \\ & + \|u\|_{X_0(t)}^p (1+t)^{-\frac{p(n-1)-n}{2}} \int_{t/2}^t (1+t-s)^{-\frac{n-2}{4}} ds. \end{aligned}$$

Due to (3.10) the term $(1+s)^{-\frac{p(n-1)-n}{2}}$ is integrable. But we have some difficulties for the treatment of the integral $\int_{t/2}^t$. If $n \le 6$ this term is not integrable. Any way we have that

$$(1+t)^{-\frac{p(n-1)-n}{2}} \int_{t/2}^t (1+t-s)^{-\frac{n-2}{4}} ds \approx \begin{cases} (1+t)^{-\frac{p(n-1)-n}{2}+1-\frac{n-2}{4}} & \text{if } n \le 5, \\ (1+t)^{-\frac{p(n-1)-n}{2}} \log(1+t) & \text{if } n = 6. \end{cases}$$

In both cases we obtain the decay thanks to (3.10). Otherwise, if $n \geq 7$ the term $(1+t-s)^{-\frac{n-2}{4}}$ is integrable. By using the fact that $p > 2$ we obtain the decay.

We notice that we can not use directly L^2-L^2 estimates because we do not have in general a good decay estimate.

Let us estimate (3.12). By using again (3.15) we have that

$$\begin{aligned}\|\partial_t^j \partial_x^\alpha Nu(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n+2(j+|\alpha|-1)}{4}} \|(\varphi,\psi)\|_{\mathcal{D}_2^{|\alpha|}} \\ + \|u\|_{X_0(t)} (1+t)^{-\frac{n+2(j+|\alpha|-1)}{4}} \int_0^{t/2} (1+s)^{-\frac{p(n-1)-n}{2}} ds \\ + \|u\|_{X_0(t)} (1+t)^{-\frac{p(n-1)-n/2}{2}} \int_0^{t/2} (1+s)^{-\frac{j+|\alpha|-1}{2}} ds\end{aligned}$$

for $(j,|\alpha|) \in \{(1,0);(0,1);(0,2)\}$.

Due to (3.10) the term $(1+s)^{-\frac{p(n-1)-n}{2}}$ is integrable. Moreover, $(j+|\alpha|-1)/2 < 1$, so we can also estimate

$$(1+t)^{-\frac{p(n-1)-n/2}{2}} \int_0^{t/2} (1+s)^{-\frac{j+|\alpha|-1}{2}} ds \approx (1+t)^{-\frac{p(n-1)-n/2}{2}+1-\frac{j+|\alpha|-1}{2}} \lesssim (1+t)^{-\frac{n+2(j+|\alpha|-1)}{4}}.$$

So the proof of (3.7) is done. Now we prove (3.8). We have that

$$\|Nu - Nv\|_{X(t)} = \left\| \int_0^t G_1(t-s,x) *_{(x)} (|u(s,x)|^p - |v(s,x)|^p)\, ds \right\|_{X(t)}.$$

So we use the fact that

$$||u|^p - |v|^p| \lesssim |u-v| \left(|u|^{p-1} + |v|^{p-1}\right). \tag{3.16}$$

By Hölder's inequality we obtain

$$\begin{aligned}\||u(s,\cdot)|^p - |v(s,\cdot)|^p\|_{L^1} \lesssim \|u(s,\cdot)-v(s,\cdot)\|_{L^p} \left(\|u(s,\cdot)\|_{L^p}^{p-1} + \|v(s,\cdot)\|_{L^p}^{p-1}\right), \\ \||u(s,\cdot)|^p - |v(s,\cdot)|^p\|_{L^2} \lesssim \|u(s,\cdot)-v(s,\cdot)\|_{L^{2p}} \left(\|u(s,\cdot)\|_{L^{2p}}^{p-1} + \|v(s,\cdot)\|_{L^{2p}}^{p-1}\right).\end{aligned}$$

In a very similar way to the previous case we divide the interval $[0,t]$ in two parts $[0,t/2]$ and $[t/2,t]$. We repeat the same argument of the previous case to estimate these two parts and after application of Gagliardo-Nirenberg inequality to the terms

$$\|u(s,\cdot)-v(s,\cdot)\|_{L^q}, \qquad \|u(s,\cdot)\|_{L^q}, \qquad \|v(s,\cdot)\|_{L^q},$$

with $q = p$ and $q = 2p$, we conclude our proof. $\square$

Remark 7.

One can expect that the more natural space for initial data is $\mathcal{D}_2^1$. But we can not repeat the above presented arguments with $\mathcal{D}_2^1$ regularity. In fact if we suppose $(\varphi, \psi) \in \mathcal{D}_2^1$, then we have that the exponent given by Gagliardo-Niremberg inequality is such that $\theta_1(q) > \theta_2(q)$, it means that we lose decay. For this reason we need H^2 instead H^1 regularity for the initial datum φ.

Chapter 4

Semi-linear wave models with mass, damping and source terms.

4.1 Main tools

Now we devote to the following Cauchy problem:

$$u_{tt} - \Delta u + m^2 u + b u_t = |u|^p,\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x), \tag{4.1}$$

where m^2 and b are two positive constants and $(\varphi, \psi) \in H^1 \times L^2$.

Our aim is to prove a global (in time) existence result.

The approach to deal with (4.1) is very similar to the one which we used in the previous chapter to treat (3.1). One tool is again Duhamel's principle, as following:

Let us consider the linear problem

$$(u_{lin})_{tt} - \Delta(u_{lin}) + m^2(u_{lin}) + b(u_{lin})_t = 0,\ (u_{lin})(0,x) = \varphi(x),\ (u_{lin})_t(0,x) = \psi(x).$$

According with Chapter 1, if we put $\alpha^2 := |m^2 - \frac{b^2}{4}|$, one has

$$(u_{lin})(t,x) = G_0(t,x) *_x \varphi(x) + G_1(t,x) *_x \psi(x);$$

being

$$F_{x\to\xi}G_0(t,\xi)=\begin{cases}\cos\Big(\sqrt{|\xi|^2+\alpha^2}\ t\Big) & \text{if } m^2-\frac{b^2}{4}\geq 0;\\ \cos\Big(\sqrt{|\xi|^2-\alpha^2}\ t\Big) & \text{if } m^2-\frac{b^2}{4}<0 \text{ and } |\xi|>\alpha;\\ \cosh\Big(\sqrt{\alpha^2-|\xi|^2}\ t\Big) & \text{if } m^2-\frac{b^2}{4}<0 \text{ and } |\xi|<\alpha.\end{cases}$$

and

$$F_{x\to\xi}G_1(t,\xi)=\begin{cases}\frac{\sin\left(\sqrt{|\xi|^2+\alpha^2}\ t\right)}{\sqrt{|\xi|^2+\alpha^2}} & \text{if } m^2-\frac{b^2}{4}\geq 0;\\ \frac{\sin\left(\sqrt{|\xi|^2-\alpha^2}\ t\right)}{\sqrt{|\xi|^2+\alpha^2}} & \text{if } m^2-\frac{b^2}{4}<0 \text{ and } |\xi|>\alpha;\\ \frac{\sinh\left(\sqrt{\alpha^2-|\xi|^2}\ t\right)}{\sqrt{|\xi|^2+\alpha^2}} & \text{if } m^2-\frac{b^2}{4}<0 \text{ and } |\xi|<\alpha.\end{cases}$$

Using Duhamel's principle any solution to (4.1) satisfies

$$u(t,x)=G_0(x,t)*_{(x)}\varphi(x)+G_1(x,t)*_{(x)}\psi(x)+\int_0^t G_1(t-s,x)*_{(x)}|u(s,x)|^p ds. \tag{4.2}$$

Having in mind Theorem 1.1.3 we define for all $t\geq 0$

$$X(t):=C([0,t],H^1)\cap C^1([0,t],L^2)$$

with the norm

$$\|w\|_{X(t)}:=\sup_{0\leq\tau\leq t}\begin{cases}e^{\frac{b}{2}\tau}\big(\|w\|_{L^2}+\|\nabla w\|_{L^2}+\|w_t\|_{L^2}\big) & \text{if } m^2-\frac{b^2}{4}>0,\\ e^{\frac{b}{2}\tau}\big((1+\tau)^{-1}\|w\|_{L^2}+\|\nabla w\|_{L^2}+(1+\tau)^{-1}\|w_t\|_{L^2}\big) & \text{if } m^2-\frac{b^2}{4}=0,\\ e^{\frac{m^2}{b}\tau}\big(\|w\|_{L^2}+(1+\tau)^{1/2}\|\nabla w\|_{L^2}+(1+\tau)\|w_t\|_{L^2}\big) & \text{if } m^2-\frac{b^2}{4}<0.\end{cases} \tag{4.3}$$

We also define

$$\|w\|_{X_0(t)}:=\sup_{0\leq\tau\leq t}\begin{cases}e^{\frac{b}{2}\tau}\big(\|w\|_{L^2}+\|\nabla w\|_{L^2}\big) & \text{if } m^2-\frac{b^2}{4}>0,\\ e^{\frac{b}{2}\tau}\big((1+\tau)^{-1}\|w\|_{L^2}+\|\nabla w\|_{L^2}\big) & \text{if } m^2-\frac{b^2}{4}=0,\\ e^{\frac{m^2}{b}\tau}\big(\|w\|_{L^2}+(1+\tau)^{1/2}\|\nabla w\|_{L^2}\big) & \text{if } m^2-\frac{b^2}{4}<0.\end{cases} \tag{4.4}$$

As in the previous case we want to prove that for any small data (φ,ψ) in the space $A:=H^1\times L^2$ the operator N which is defined for any $w\in X(t)$ by

$$N:\ w\to Nw:=G_0(x,t)*_{(x)}\varphi(x)+G_1(x,t)*_{(x)}\psi(x)+\int_0^t G_1(t-s,x)*_{(x)}|w(s,x)|^p ds$$

satisfies the following estimates:

$$\|Nu\|_{X(t)} \lesssim \|(\varphi,\psi)\|_{H^1\times L^2} + \|u\|^p_{X_0(t)}, \tag{4.5}$$

$$\|Nu - Nv\|_{X(t)} \lesssim \|u-v\|_{X_0(t)} \left(\|u\|^{p-1}_{X_0(t)} + \|v\|^{p-1}_{X_0(t)}\right) \tag{4.6}$$

uniformly with respect to $t \geq 0$. As in the previous case from (4.5) and (4.6) it follows that for small initial data N maps $X(t)$ into itself and that N has a unique fixed point u which is also the unique solution to (4.1). For this aim we will use Gagliardo-Nirenberg inequality from Theorem 3.1.2.

Finally we notice that due to the definition of $X(t)$ is chosen in an appropriate way to obtain the decay estimates for the solution to the semi-linear problem, too.

4.2 Global existence and decay behaviour

Theorem 4.2.1.
Let $A := H^1 \times L^1$, $n \geq 2$ and let $p \in (1,\ n/(n-2)]$. Then there exists $\varepsilon > 0$ such that for any $(\varphi,\psi) \in H^1 \times L^2$ with $\|(\varphi,\psi)\|_{H^1\times L^2} < \varepsilon$ there exists a unique solution $u \in C([0,\infty), H^1) \cap C^1([0,\infty), L^2)$ to (3.1).

Moreover the solution, its first derivatives in space and its first derivative in time satisfy the decay estimates (1.2), (1.3) and (1.4).

Proof.
As we explained before we just need to prove (4.5) and (4.6). We should distinguish three cases $m^2 - \frac{b^2}{4} > 0$, $m^2 - \frac{b^2}{4} = 0$ and $m^2 - \frac{b^2}{4} < 0$. But the treatment of the first and second cases is very similar to the third one. So, henceforth, we will restrict ourselves to the case $m^2 - \frac{b^2}{4} < 0$.

Let us prove (4.5). By using (1.2), (1.3) and (1.4) we have that

$$\|Nu(t,\cdot)\|_{L^2} \lesssim e^{-\frac{m^2}{b}t}\|(\varphi,\psi)\|_{H^1\times L^2} + \int_0^t e^{-\frac{m^2}{b}(t-s)}\||u(s,\cdot)|^p\|_{L^2}ds; \tag{4.7}$$

$$\|\nabla Nu(t,\cdot)\|_{L^2} \lesssim (1+t)^{-1/2}e^{-\frac{m^2}{b}t}\|(\varphi,\psi)\|_{H^1\times L^2} + \int_0^t (1+t-s)^{-1/2}e^{-\frac{m^2}{b}(t-s)}\||u(s,\cdot)|^p\|_{L^2}ds; \tag{4.8}$$

$$\|\partial_t Nu(t,\cdot)\|_{L^2} \lesssim (1+t)^{-1}e^{-\frac{m^2}{b}t}\|(\varphi,\psi)\|_{H^1\times L^2} + \int_0^t (1+t-s)^{-1}e^{-\frac{m^2}{b}(t-s)}\||u(s,\cdot)|^p\|_{L^2}ds. \tag{4.9}$$

In a very similar way to the previous case and by using (3.9) with $q=2p$ and $k=1$ we can conclude that

$$\||u(s,\cdot)|^p\|_{L^2} \lesssim \|u\|^p_{X_0(t)}e^{-p\frac{m^2}{b}s}(1+s)^{-\frac{p\theta(2p)}{2}} \lesssim \|u\|^p_{X_0(t)}e^{-p\frac{m^2}{b}s}(1+s)^{-\frac{n(p-1)}{4}}. \tag{4.10}$$

In the integral we can estimate as follows:

$$\begin{aligned}\int_0^t &(1+t-s)^{-\frac{\alpha}{2}}e^{-\frac{m^2}{b}(t-s)}(1+s)^{-\frac{n(p-1)}{4}}e^{-p\frac{m^2}{b}s}ds\\ &\lesssim e^{-\frac{m^2}{b}t}(1+t)^{-\frac{\alpha}{2}}\int_0^{t/2}(1+s)^{-\frac{n(p-1)}{4}}e^{(1-p)\frac{m^2}{b}s}ds\\ &\quad + e^{-\frac{m^2}{b}t}(1+t)^{-\frac{\alpha}{2}}\int_{t/2}^t(1+t-s)^{-\frac{\alpha}{2}}(1+s)^{-\frac{n(p-1)-2\alpha}{4}}e^{(1-p)\frac{m^2}{b}s}ds\\ &\lesssim e^{-\frac{m^2}{b}t}(1+t)^{-\frac{\alpha}{2}}\int_0^t(1+s)^{-\frac{n(p-1)-2\alpha}{4}}e^{(1-p)\frac{m^2}{b}s}ds,\end{aligned} \tag{4.11}$$

where $\alpha = 0,1,2$.

But $p>1$. So we have that $(1+s)^{-\frac{n(p-1)-2\alpha}{4}+\frac{\alpha}{2}}e^{(1-p)\frac{m^2}{b}s}$ is integrable. The proof to show (4.5) is completed.

Let us derive (4.6). We notice that

$$\|Nu-Nv\|_{X(t)} \lesssim \left\|\int_0^t G_1(t-s,x)*_{(x)}\left(|u(s,x)|^p-|v(s,x)|^p\right)ds\right\|_{X(t)}.$$

Thanks to (1.2), (1.3) and (1.4) we can estimate as follow:

$$\|\nabla^j\partial_t^\ell G_1(t-s,x)*_{(x)}\left(|u(s,x)|^p-|v(s,x)|^p\right)\|_{L^2} \lesssim e^{-\frac{m^2}{b}(t-s)}(1+t-s)^{-\ell-\frac{j}{2}}\||u(s,x)|^p-|v(s,x)|^p\|_{L^2}ds, \tag{4.12}$$

where $\ell + j = 0, 1$. By using (3.16) and Hölder's inequality one can estimate as follows:

$$\||u(s,\cdot)|^p - |v(s,\cdot)|^p\|_{L^2} \lesssim \|u(s,\cdot) - v(s,\cdot)\|_{L^{2p}} \left(\|u(s,\cdot)\|_{L^{2p}}^{p-1} + \|v(s,\cdot)\|_{L^{2p}}^{p-1}\right).$$

Analogously to the proof of (4.5) we apply Gagliardo-Nirenberg inequality and obtain

$$\|u(s,\cdot) - v(s,\cdot)\|_{L^{2p}} \lesssim e^{-\frac{m^2}{b}s}(1+s)^{-\frac{n}{4}\frac{p-1}{p}}\|u-v\|_{X_0(t)},$$
$$\|u(s,\cdot)\|_{L^{2p}}^{p-1} \lesssim e^{-\frac{m^2}{b}s(p-1)}(1+s)^{-\frac{n}{4}\frac{(p-1)^2}{p}}\|u\|_{X_0(t)}^{p-1}.$$

Summarizing brings

$$\|u(s,\cdot) - v(s,\cdot)\|_{L^{2p}} \lesssim \|u-v\|_{X_0(t)} \left(\|u\|_{X_0(t)}^{p-1} + \|v\|_{X_0(t)}^{p-1}\right)(1+s)^{-\frac{n}{4}(p-1)}e^{-\frac{m^2}{b}ps}. \tag{4.13}$$

So, if $j + l = 0, 1$, then due to (4.13) it holds, by using the same reasoning of (4.11) we obtain

$$\begin{aligned}
&\|\nabla^j \partial_t^\ell (Nu - Nv)\|_{L^2} \\
&\quad \lesssim \|u-v\|_{X_0(t)} \left(\|u\|_{X_0(t)}^{p-1} + \|v\|_{X_0(t)}^{p-1}\right) e^{-\frac{m^2}{b}t} \int_0^t (1+t-s)^{-\ell-\frac{j}{2}}(1+s)^{-\frac{n}{4}(p-1)} e^{\frac{m^2}{b}(1-p)s} \\
&\qquad \lesssim \|u-v\|_{X_0(t)} \left(\|u\|_{X_0(t)}^{p-1} + \|v\|_{X_0(t)}^{p-1}\right) e^{-\frac{m^2}{b}t}(1+t)^{-\ell-\frac{j}{2}}.
\end{aligned}$$

and this concludes the proof of (4.6). □

Remark 8.
In this model we need only $L^2 - L^2$ estimates because of the exponential decay estimates for the linear model. If $m \to 0$, then we lose the exponential decay. In this case one needs $L^2 \cap L^1 - L^2$ estimates to obtain a suitable decay. This model has already been studied in [1] where the damping coeffincient b can depends also on time.

Chapter 5

Visco-elastic damped wave models with power non-linearity $|u_t|^p$.

5.1 Main tools

Now we study the Cauchy problem for the semi-linear visco-elastic damped wave model of the form

$$u_{tt} - \Delta u - \Delta u_t = |u_t|^p,\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x), \tag{5.1}$$

where $(\varphi, \psi) \in (H^s \cap L^1) \times (H^{s-2} \cap L^1)$ for a certain s that we will specify later.

Our aim is to prove a global (in time) existence result.

The approach to deal (5.1) is similar to the one which we used to treat (3.1) and (4.1). One tool is again Duhamel's principle, as following:

Let us consider the linear problem

$$(u_{lin})_{tt} - \Delta(u_{lin}) - \Delta(u_{lin})_t = 0,\ (u_{lin})(0,x) = \varphi(x),\ (u_{lin})_t(0,x) = \psi(x).$$

According with Chapter 2 one has

$$(u_{lin})(t,x) = G_0(t,x) *_x \varphi(x) + G_1(t,x) *_x \psi(x);$$

being G_0 defined by (3.2) and G_1 defined by (3.3).

Using Duhamel's principle any solution to (5.1) satisfies

$$u(t,x) = G_0(x,t) *_{(x)} \varphi(x) + G_1(x,t) *_{(x)} \psi(x) + \int_0^t G_1(t-\tau,x) *_{(x)} |u_t(\tau,x)|^p d\tau. \tag{5.2}$$

Having in mind Theorem 2.1.4 for a certain $s \geq 0$ which will be specified later we define for all $t \geq 0$ the function space

$$X(t) := C([0,t], H^s) \cap C^1([0,t], H^{s-2})$$

with the norm

$$\|w\|_{X(t)} := \sup_{0\leq\tau\leq t}\Big((1+\tau)^{\frac{n-2}{4}}\|w\|_{L^2} + (1+\tau)^{\frac{n+2(s-1)}{4}}\||D|^s w\|_{L^2} + (1+\tau)^{\frac{n}{4}}\|w_t\|_{L^2} + (1+\tau)^{\frac{n+2(s-2)}{4}}\||D|^{s-2}w_t\|_{L^2}\Big). \tag{5.3}$$

More precisely, as in the previous case, if $n=2$, then we have

$$\|w\|_{X(t)} := \sup_{0\leq\tau\leq t}\Big((\log(e+t))^{-1}\|w\|_{L^2} + (1+\tau)^{\frac{n+2(s-1)}{4}}\||D|^s w\|_{L^2} + (1+\tau)^{\frac{n}{4}}\|w_t\|_{L^2} + (1+\tau)^{\frac{n+2(s-2)}{4}}\||D|^{s-2}w_t\|_{L^2}\Big),$$

the term $(\log(e+t))^{-1}$ brings no additional difficulties so we will ignore it. It is easy to prove that $\big(X(t), \|\cdot\|_{X(t)}\big)$ is a Banach space.

Our aim is to prove that for any data (φ,ψ) from the space $A := \big(H^s \cap L^1\big) \times \big(H^{s-2} \cap L^1\big)$ the operator N which is defined for any $w \in X(t)$ by

$$N:\ w \ \to\ Nw := G_0(x,t) *_{(x)} \varphi(x) + G_1(x,t) *_{(x)} \psi(x) + \int_0^t G_1(t-\tau,x) *_{(x)} |w_t(\tau,x)|^p d\tau$$

satisfies the following estimates:

$$\|Nu\|_{X(t)} \lesssim \|(\varphi,\psi)\|_A + \|u\|^p_{X(t)}, \tag{5.4}$$

$$\|Nu - Nv\|_{X(t)} \lesssim \|u-v\|_{X(t)}\Big(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)}\Big) \tag{5.5}$$

uniformly with respect to $t \geq 0$. As we explained before, after showing (5.4) and (5.5) we may conclude that for small initial data the operator N has a unique fixed point which is also the unique solution of (5.1). Finally we notice that due to the definition of $X(t)$ is chosen in an appropriate way to obtain the decay estimates for the solution to the semi-linear problem, too.

The strategy is quite similar to the proof of Theorem 3.2.1 but we need some others tools. As in the proof of Theorem 3.2.1 we will use Gagliardo-Nirenberg inequality. Nevertheless Theorem 3.1.2 is not sufficient for our approach. For this reason we introduce the following results:

Theorem 5.1.1 (Generalized Gagliardo-Nirenberg inequality)**.**
Let $a \in [0, \sigma)$. Then for all $u \in H^{\sigma,m}$ we have the following results for $m \in (1, \infty)$:

$$\| |D|^a u\|_{L^q} \lesssim \| |D|^\sigma u\|_{L^m}^{\theta_{a,\sigma}(q,m)} \|u\|_{L^m}^{1-\theta_{a,\sigma}(q,m)}, \tag{5.6}$$

where $\frac{a}{\sigma} \leq \theta_{a,\sigma}(q,m) < 1$, *and* $\theta_{a,\sigma}(q,m) = \frac{n}{\sigma}\left(\frac{1}{m} - \frac{1}{q} + \frac{a}{n}\right)$, *hence* $m \leq q < \frac{nm}{n + m(a-\sigma)}$.

For the proof one can see [6] and [3].

We will also use a superposition result from [8].

Proposition 5.1.2.
Let $p > 1$ and $v \in H^{s,m}$, where $s \in \left(\frac{n}{p}, p\right)$. Then the following estimate holds:

$$\| |v|^p\|_{H^{s,m}} \lesssim \|v\|_{H^{s,m}} \|v\|_{L^\infty}^{p-1}. \tag{5.7}$$

We use the following corollary from Proposition 5.1.2.

Corollary 5.1.3.
Under the assumptions of Proposition 5.1.2 it holds

$$\| |v|^p\|_{\dot{H}^{s,m}} \lesssim \|v\|_{\dot{H}^{s,m}} \|v\|_{L^\infty}^{p-1}. \tag{5.8}$$

For the proof one can see [3].

5.2 Global existence and decay behaviour

Theorem 5.2.1.
Let $A := (H^{s+2} \cap L^1) \times (H^s \cap L^1)$. Let us consider the Cauchy problem (5.1) with initial data $(\varphi, \psi) \in A$. Let

$$s > 2 + \frac{n}{2}, \text{ where } n \geq 2. \tag{5.9}$$

Then for any $p > s$ there exists a uniquely determined global (in time) small data solution belonging to $C([0,\infty), H^s) \cap C^1([0,\infty), H^{s-2})$.

Moreover u, u_t, $|D|^s u$ and $|D|^{s-2} u_t$ satisfy the following decay estimates:

$$\|u(t,\cdot)\|_{L^2} \lesssim \begin{cases} (1+t)^{-\frac{n-2}{4}} \|(\varphi,\psi)\|_A & \text{if } n \geq 3; \\ \log(e+t)\|(\varphi,\psi)\|_A & \text{if } n = 2; \end{cases}$$
$$\|u_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{4}} \|(\varphi,\psi)\|_A$$
$$\||D|^{s+2} u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n+2(s+1)}{4}} \|(\varphi,\psi)\|_A;$$
$$\||D|^{s} u_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n+2s}{4}} \|(\varphi,\psi)\|_A.$$

Proof.
Our aim is to prove the two inequalities (5.4) and (5.5). Let us prove the first inequality. Now we estimate the L^2 norm of Nu. Thanks to Theorems 2.1.4 and 2.1.6 we have that

$$\begin{aligned}\|Nu(t,\cdot)\|_{L^2} &\lesssim (1+t)^{-\frac{n-2}{4}} \|(\varphi,\psi)\|_A + \left\| \int_0^t G_1(t-\tau,x) *_{(x)} |u_t(\tau,x)|^p d\tau \right\|_{L^2} \\ &\lesssim (1+t)^{-\frac{n-2}{4}} \|(\varphi,\psi)\|_A + \int_0^t (1+t-\tau)^{-\frac{n-2}{4}} \||u_t(\tau,x)|^p\|_{L^2\cap L^1} d\tau.\end{aligned}$$

We have $\||u_t(\tau,x)|^p\|_{L^2\cap L^1} \lesssim \|u_t\|^p_{L^p} + \|u_t\|^p_{L^{2p}}$. To estimate $\|u_t\|^p_{L^{kp}}$, $k = 1, 2$, we use (5.6) with $a = 0$, $\sigma = s-2$, $q = kp$ and $m = 2$. We get

$$\|u_t\|_{L^{kp}} \leq \||D|^{s-2} u_t\|_{L^2}^{\theta_{0,s-2}(kp,2)} \|u_t\|_{L^2}^{1-\theta_{0,s-2}(kp,2)}, \tag{5.10}$$

where $\theta_{0,s-2}(kp,2) = \dfrac{n}{s-2}\left(\dfrac{1}{2} - \dfrac{1}{kp}\right) \in [0,1)$ as in Corollary 5.1.3, that is

$$2 \leq kp < \frac{2n}{n-2(s-2)} \quad \text{or} \quad 2 \leq kp \;\text{ if }\; \frac{n}{2(s-2)} \leq 1.$$

Because of (5.9) we have $\dfrac{n}{2(s-2)} < 1$ and so it is sufficient to choose $p > 2$.

Since $\theta_{0,s-2}(p,2) < \theta_{0,s-2}(2p,2)$ by using (2.7) and (2.18) with $m = 2$ and $\beta = s-2$ we have that

$$\begin{aligned}\| |u_t(\tau,x)|^p\|_{L^2\cap L^1} d\tau &\lesssim \|u\|^p_{X(\tau)}(1+\tau)^{-\frac{n+2(s-2)}{4}p\theta}(1+\tau)^{-\frac{n}{4}p(1-\theta)}\\ &\lesssim \|u\|^p_{X(\tau)}(1+\tau)^{-\frac{n}{2}(p-1)},\end{aligned}$$

where $\theta = \theta_{0,s-2}(p,2)$. By using (3.15) we can conclude that

$$\begin{aligned}\int_0^t (1+t-\tau)^{-\frac{n-2}{4}}\| |u_t(\tau,x)|^p\|_{L^2\cap L^1} d\tau \leq \|u\|^p_{X(t)}\Big((1+t)^{-\frac{n-2}{4}}\int_0^{t/2}(1+\tau)^{-\frac{n}{2}(p-1)}d\tau\\ + \int_{t/2}^t (1+t-\tau)^{-\frac{n-2}{4}}(1+\tau)^{-\frac{n}{2}(p-1)}d\tau\Big).\end{aligned}$$

Let us examine the integral $\int_0^{t/2}$. Due to (5.9) and $p > s$ we have $\frac{n}{2}(p-1) > 1$. Here we exclude $n = 1$. The term $(1+\tau)^{-\frac{n}{2}(p-1)}$ is integrable. For the integral $\int_{t/2}^t$ one can repeat the reasoning for estimating Nu in the semi-linear visco-elastic model with source term and then we may conclude that

$$(1+t)^{\frac{n-2}{4}}\|Nu(t,\cdot)\|_{L^2} \lesssim \|(\varphi,\psi)\|_A + \|u\|^p_{X(t)}. \tag{5.11}$$

Let us estimate the term $\partial_t Nu$. We will divide the interval $[0,t]$ in two sub-intervals $[0,t/2]$ and $[t/2,t]$. In the first one we need $L^2\cap L^1 - L^2$ estimates and in the second one we need $L^2 - L^2$ estimates. We will use inequalities (2.7) and (2.18) for the first interval and inequalities (2.11) and (2.20) for the second one. Then in a very similar way to the previous case

$$\begin{aligned}\|\partial_t Nu(t,\cdot)\|_{L^2} \lesssim&(1+t)^{-\frac{n}{4}}\|(\varphi,\psi)\|_A + \|u\|^p_{X(t)}\Big(\int_0^{t/2}(1+t-\tau)^{-\frac{n}{4}}(1+\tau)^{-\frac{n}{2}(p-1)}d\tau\\ &+ \int_{t/2}^t (1+\tau)^{-\frac{n}{4}(2p-1)}d\tau\Big).\end{aligned}$$

To treat the integral $\int_0^{t/2}$ we repeat the approach of the previous case. Due to $p > s$ and (5.9) we have

$$\int_0^{t/2}(1+t-\tau)^{-\frac{n}{4}}(1+\tau)^{-\frac{n}{2}(p-1)}d\tau \lesssim (1+t)^{-\frac{n}{4}}.$$

For the integral $\int_{t/2}^t$ we estimate as follows:

$$\int_{t/2}^t (1+\tau)^{-\frac{n}{4}(2p-1)}d\tau \lesssim (1+t)^{-\frac{n}{4}}\int_{t/2}^t (1+\tau)^{-\frac{n}{2}(p-1)}d\tau \lesssim (1+t)^{-\frac{n}{4}}.$$

Summarizing we proved that

$$(1+t)^{\frac{n}{4}}\|\partial_t Nu(t,\cdot)\|_{L^2} \lesssim \|(\varphi,\psi)\|_A + \|u\|^p_{X(t)}. \tag{5.12}$$

Now let us estimate the term $\|\partial_t|D|^{s-2}Nu\|_{L^2}$. We will proceed as above. We will divide the interval $[0,t]$ in two sub-intervals $[0,t/2]$ and $[t/2,t]$. We will use $L^2\cap L^1 - L^2$ estimates for the first sub-interval and L^2-L^2 estimates for the second one. So by using (2.18) and (2.20) we have

$$\begin{aligned}
\|\partial_t|D|^{s-2}Nu(t,\cdot)\|_{L^2} &\lesssim (1+t)^{-\frac{n+2(s-2)}{4}}\|(\varphi,\psi)\|_A \\
&\quad + \int_0^{t/2}(1+t-\tau)^{-\frac{n+2(s-2)}{4}}\||u_t|^p\|_{H^{s-2}\cap L^1}d\tau \\
&\quad + \int_{t/2}^t(1+t-\tau)^{-\frac{s-2}{2}}\||u_t|^p\|_{H^{s-2}}d\tau \\
&\lesssim (1+t)^{-\frac{n+2(s-2)}{4}}\|(\varphi,\psi)\|_A \\
&\quad + \int_0^{t/2}(1+t-\tau)^{-\frac{n+2(s-2)}{4}}\left(\||u_t|^p\|_{L^2\cap L^1} + \||u_t|^p\|_{\dot H^{s-2}}\right)d\tau \\
&\quad + \int_{t/2}^t(1+t-\tau)^{-\frac{s-2}{2}}\left(\||u_t|^p\|_{L^2} + \||u_t(\tau,\cdot)|^p\|_{\dot H^{s-2}}\right)d\tau.
\end{aligned}$$

The integrals with $\||u_t|^p\|_{L^2\cap L^1}$ or $\||u_t|^p\|_{L^2}$ will be handled as before if we apply $p>s$ and (5.9). Let us estimate the term $\||u_t|^p\|_{\dot H^{s-2}}$. Then by Proposition 5.1.2 we have

$$\begin{aligned}
\||u_t|^p(\tau,\cdot)\|_{\dot H^{s-2}} &\lesssim \|u_t\|_{\dot H^{s-2}}\|u_t\|^{p-1}_{L^\infty} \lesssim \|u_t\|_{\dot H^{s-2}}\|u_t\|^{p-1}_{H^{s_0}} \lesssim \|u_t\|_{\dot H^{s-2}}\left(\|u_t\|_{L^2} + \|u_t\|_{\dot H^{s_0}}\right)^{p-1} \\
&\lesssim \|u\|^p_{X(\tau)}(1+\tau)^{-\frac{n+2(s-2)}{4}}\left((1+\tau)^{-\frac{n}{4}(p-1)} + (1+\tau)^{-\frac{n+2s_0}{4}(p-1)}\right) \\
&\lesssim \|u\|^p_{X(\tau)}\left((1+\tau)^{-\frac{np+2(s-2)}{4}} + (1+\tau)^{-\frac{n+2(s-2)}{4}p}\right) \\
&\lesssim \|u\|^p_{X(t)}(1+\tau)^{-\frac{np+2(s-2)}{4}},
\end{aligned} \tag{5.13}$$

where $s-2>s_0>\dfrac{n}{2}$. Using (5.9) and $p>s$ implies $\frac{np+2(s-2)}{4}>1$. Hence, it holds

$$\begin{aligned}
\int_0^{t/2}(1+t-\tau)^{-\frac{n+2(s-2)}{4}}\||u_t|^p\|_{\dot H^{s-2}}d\tau &\lesssim (1+t)^{-\frac{n+2(s-2)}{4}}\|u\|^p_{X(t)}\int_0^{t/2}(1+\tau)^{-\frac{np+2(s-2)}{4}}d\tau \\
&\lesssim (1+t)^{-\frac{n+2(s-2)}{4}}\|u\|^p_{X(t)}.
\end{aligned}$$

In an analogous way we estimate

$$\int_{t/2}^t(1+t-\tau)^{-\frac{s-2}{2}}\||u_t|^p\|_{\dot H^{s-2}}d\tau \lesssim (1+t)^{-\frac{n+2(s-2)}{4}}\|u\|^p_{X(t)}\int_{t/2}^t(1+t-\tau)^{-\frac{s+2}{2}}(1+\tau)^{-\frac{n(p-1)}{4}}d\tau.$$

On $[t/2,t]$ we have that

$$(1+t-\tau)^{-\frac{s-2}{2}}(1+\tau)^{-\frac{n(p-1)}{4}} \lesssim (1+t-\tau)^{-\frac{n(p-1)+2(s-2)}{4}}$$

and due to (5.9) and $p>s$ we conclude $\frac{n(p-1)+2(s-2)}{4}>1$. Summarizing we have shown that

$$(1+t)^{\frac{n+2(s-2)}{4}}\|\partial_t|D|^{s-2}Nu(t,\cdot)\|_{L^2} \lesssim \|(\varphi,\psi)\|_A + \|u\|^p_{X(t)}. \tag{5.14}$$

By using the same approach one can prove that

$$(1+t)^{\frac{n+2(s-1)}{4}}\||D|^{s}Nu(t,\cdot)\|_{L^2} \lesssim \|(\varphi,\psi)\|_A + \|u\|^p_{X(t)}. \tag{5.15}$$

Summarizing (5.12), (5.11), (5.14) and (5.15) we have proved (5.4).

Let us prove (5.5). We will use the same approach as in the previous case. We know that

$$\|Nu-Nv\|_{X(t)} \lesssim \left\| \int_0^t G_1(t-s,x) *_{(x)} \left(|u_t(s,x)|^p - |v_t(s,x)|^p\right) ds \right\|_{X(t)}.$$

Hence, by using (2.5) we have

$$\|Nu(t,\cdot)-Nv(t,\cdot)\|_{L^2} \lesssim \int_0^t (1+t-\tau)^{-\frac{n-2}{4}} \||u_t(\tau,x)|^p - |v_t(\tau,x)|^p\|_{L^2\cap L^1} d\tau.$$

Due to (3.16) it holds

$$\begin{aligned}
\||u_t(\tau,\cdot)|^p - |v_t(\tau,\cdot)|^p\|_{L^1} &\lesssim \|u_t(\tau,\cdot)-v_t(\tau,\cdot)\|_{L^p}\left(\|u_t(\tau,\cdot)\|^{p-1}_{L^p} + \|v_t(\tau,\cdot)\|^{p-1}_{L^p}\right);\\
\||u_t(\tau,\cdot)|^p - |v_t(\tau,\cdot)|^p\|_{L^2} &\lesssim \|u_t(\tau,\cdot)-v_t(\tau,\cdot)\|_{L^{2p}}\left(\|u_t(\tau,\cdot)\|^{p-1}_{L^{2p}} + \|v_t(\tau,\cdot)\|^{p-1}_{L^{2p}}\right).
\end{aligned}$$

By using Gagliardo-Nirenberg inequality we have that

$$\begin{aligned}
\|u_t(\tau,\cdot)-v_t(\tau,\cdot)\|_{L^{kp}} &\lesssim \||D|^{s-2}u_t - |D|^{s-2}v_t\|^{\theta}_{L^2}\|u_t-v_t\|^{1-\theta}_{L^2},\\
\|u_t(\tau,\cdot)\|^{p-1}_{L^{kp}} &\lesssim \||D|^{s-2}u_t\|^{\theta(p-1)}_{L^2}\|u_t\|^{(1-\theta)(p-1)}_{L^2},
\end{aligned}$$

where $\theta=\theta_{0,s-2}(kp,2)=\dfrac{n}{s-2}\left(\dfrac{1}{2}-\dfrac{1}{kp}\right)$ and $k=1,2$. So we obtain the estimates

$$\|u_t(\tau,\cdot)-v_t(\tau,\cdot)\|_{L^{kp}} \lesssim \|u-v\|_{X(\tau)}(1+\tau)^{-\frac{n}{2kp}(kp-1)}, \tag{5.16}$$

$$\|u_t(\tau,\cdot)\|^{p-1}_{L^{kp}} \lesssim \|u\|_{X(\tau)}(1+\tau)^{-\frac{n}{2kp}(kp-1)(p-1)}. \tag{5.17}$$

Consequently,

$$\||u_t(\tau,\cdot)|^p - |v_t(\tau,\cdot)|^p\|_{L^k} \lesssim \|u-v\|_{X(\tau)} \left(\|u\|^{p-1}_{X(\tau)} + \|v\|^{p-1}_{X(\tau)}\right)(1+\tau)^{-\frac{n}{2k}(kp-1)}. \tag{5.18}$$

Since $\theta_{0,s-2}(p,2) \leq \theta_{0,s-2}(2p,2)$ we may conclude

$$\|Nu - Nv\|_{L^2} \lesssim \|u-v\|_{X(t)} \left(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)}\right) \int_0^t (1+t-\tau)^{-\frac{n-2}{4}}(1+\tau)^{-\frac{n}{2}(p-1)} d\tau.$$

We should estimate the integral but we have already estimated this term when we proved (5.11). So we may conclude

$$(1+t)^{-\frac{n-2}{4}}\|Nu(t,\cdot) - Nv(t,\cdot)\|_{L^2} \lesssim \|u-v\|_{X(t)} \left(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)}\right). \tag{5.19}$$

Let us estimate the term $\partial_t(Nu - Nv)$. We will divide the interval $[0,t]$ in two sub-intervals $[0,t/2]$ and $[t/2,t]$. In the first one we need $L^2 \cap L^1 - L^2$ estimates and in the second one we need $L^2 - L^2$ estimates. We will use inequalities (2.7) and (2.18) for the first interval and inequalities (2.11) and (2.20) for the second one. As above by using (5.18) we have

$$\begin{aligned}
\|\partial_t(Nu - Nv)\|_{L^2} &\lesssim \int_0^{t/2} (1+t-\tau)^{-\frac{n}{4}} \||u_t(\tau,x)|^p - |v_t(\tau,x)|^p\|_{L^2\cap L^1} d\tau \\
&\quad + \int_{t/2}^t \||u_t(\tau,x)|^p - |v_t(\tau,x)|^p\|_{L^2} d\tau \\
&\lesssim \|u-v\|_{X(t)} \left(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)}\right) \left[\int_0^{t/2} (1+t-\tau)^{-\frac{n}{4}}(1+\tau)^{-\frac{n}{2}(p-1)} d\tau \right. \\
&\quad \left. + \int_{t/2}^t (1+\tau)^{-\frac{n}{4}(2p-1)} d\tau \right].
\end{aligned}$$

To estimate the integrals we repeat the reasoning of the proof of (5.12). So we may conclude

$$(1+t)^{-\frac{n}{4}}\|\partial_t(Nu - Nv)(\tau,\cdot)\|_{L^2} \lesssim \|u-v\|_{X(t)} \left(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)}\right). \tag{5.20}$$

Now we estimate the term $\partial_t|D|^{s-2}(Nu - Nv)$. We will divide the interval $[0,t]$ in two sub-intervals $[0,t/2]$ and $[t/2,t]$. In the first one we need $L^2 \cap L^1 - L^2$ estimates and in the second one we need $L^2 - L^2$ estimates. By using a similar approach to the proof of (5.4) we have

$$\begin{aligned}
\|\partial_t|D|^{s-2}(Nu - Nv)\|_{L^2} &\lesssim \int_0^{t/2} (1+t-\tau)^{-\frac{n+2(s-2)}{4}} \||u_t(\tau,x)|^p - |v_t(\tau,x)|^p\|_{H^{s-2}\cap L^1} d\tau \\
&\quad + \int_{t/2}^t (1+t-\tau)^{-\frac{s-2}{2}} \||u_t(\tau,x)|^p - |v_t(\tau,x)|^p\|_{H^{s-2}} d\tau.
\end{aligned}$$

Taking account of

$$\||u_t(\tau,\cdot)|^p-|v_t(\tau,\cdot)|^p\|_{H^{s-2}\cap L^1}\lesssim \||u_t(\tau,\cdot)|^p-|v_t(\tau,\cdot)|^p\|_{\dot H^{s-2}}+\||u_t(\tau,\cdot)|^p-|v_t(\tau,\cdot)|^p\|_{L^2\cap L^1},$$
$$\||u_t(\tau,\cdot)|^p-|v_t(\tau,\cdot)|^p\|_{H^{s-2}}\lesssim \||u_t(\tau,\cdot)|^p-|v_t(\tau,\cdot)|^p\|_{\dot H^{s-2}}+\||u_t(\tau,\cdot)|^p-|v_t(\tau,\cdot)|^p\|_{L^2}.$$

The integrals with $\||u_t(\tau,\cdot)|^p-|v_t(\tau,\cdot)|^p\|_{L^2\cap L^1}$ or $\||u_t(\tau,\cdot)|^p-|v_t(\tau,\cdot)|^p\|_{L^2}$ will be handled as before. Now let us estimate the integrals with $\||u_t(\tau,\cdot)|^p-|v_t(\tau,\cdot)|^p\|_{\dot H^{s-2}}$. By using (3.16) and (5.8) we have that

$$\begin{aligned}\||u_t(\tau,\cdot)|^p-|v_t(\tau,\cdot)|^p\|_{\dot H^{s-2}}&\lesssim \|u_t-v_t\|_{\dot H^{s-2}}\left(\|u_t\|^{p-1}_{L^\infty}+\|v_t\|^{p-1}_{L^\infty}\right)\\&\lesssim \|u_t-v_t\|_{\dot H^{s-2}}\left(\|u_t\|^{p-1}_{H^{s_0}}+\|v_t\|^{p-1}_{H^{s_0}}\right),\end{aligned}$$

where $s-2>s_0>\dfrac{n}{2}$. By using the same reasoning as in the previous case implies

$$\|u_t(\tau,\cdot)-v_t(\tau,\cdot)\|_{\dot H^{s-2}}\lesssim (1+\tau)^{-\frac{n+2(s-2)}{4}}\|u(\tau,\cdot)-v(\tau,\cdot)\|_{X(\tau)},\tag{5.21}$$

$$\|u_t(\tau,\cdot)\|^{p-1}_{H^{s_0}}\lesssim \|u_t(\tau,\cdot)\|^{p-1}_{\dot H^{s_0}}+\|u_t(\tau,\cdot)\|^{p-1}_{L^2}\lesssim (1+\tau)^{-\frac{n}{4}(p-1)}\|u\|^{p-1}_{X(t)}.\tag{5.22}$$

Due to (5.21) and (5.22) we estimate

$$\begin{aligned}&\int_0^{t/2}(1+t-\tau)^{-\frac{n+2(s-2)}{4}}\||u_t(\tau,\cdot)|^p-|v_t(\tau,\cdot)|^p\|_{\dot H^{s-2}}d\tau\\&\qquad\lesssim (1+t)^{-\frac{n+2(s-2)}{4}}\|u-v\|_{X(t)}\left(\|u\|^{p-1}_{X(t)}+\|v\|^{p-1}_{X(t)}\right)\int_0^{t/2}(1+\tau)^{-\frac{np+2(s-2)}{4}}d\tau,\end{aligned}$$

and

$$\begin{aligned}&\int_{t/2}^{t}(1+t-\tau)^{-\frac{s-2}{2}}\||u_t(\tau,\cdot)|^p-|v_t(\tau,\cdot)|^p\|_{\dot H^{s-2}}d\tau\\&\qquad\lesssim (1+t)^{-\frac{n+2(s-2)}{4}}\|u-v\|_{X(t)}\left(\|u\|^{p-1}_{X(t)}+\|v\|^{p-1}_{X(t)}\right)\int_{t/2}^{t}(1+t-\tau)^{-\frac{s-2}{2}}(1+\tau)^{-\frac{n(p-1)}{4}}d\tau.\end{aligned}$$

Due to (5.9), $p>s$ and summarizing the estimates for the two integrals gives

$$(1+t)^{\frac{n+2(s-2)}{4}}\|\partial_t|D|^{s-2}(Nu-Nv)(t,\cdot)\|_{L^2}\lesssim \|u-v\|_{X(t)}\left(\|u\|^{p-1}_{X(t)}+\|v\|^{p-1}_{X(t)}\right).\tag{5.23}$$

Applying the same approach one can prove

$$(1+t)^{\frac{n+2(s-1)}{4}}\||D|^{s}(Nu-Nv)(t,\cdot)\|_{L^2}\lesssim \|u-v\|_{X(t)}\left(\|u\|^{p-1}_{X(t)}+\|v\|^{p-1}_{X(t)}\right).\tag{5.24}$$

Summarizing (5.19), (5.20), (5.23) and (5.24) we have proved (5.5) and this completes the proof. □

Chapter 6

Visco-elastic damped wave models with power non-linearity $||D|^a u|^p$ and $a \in (0,2)$.

6.1 Main tools

Now we study the Cauchy problem for another semi-linear visco-elastic damped wave model of the form

$$u_{tt} - \Delta u - \Delta u_t = ||D|^a u|^p,\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x), \tag{6.1}$$

where $(\varphi, \psi) \in (H^s \cap L^1) \times (H^{s-2} \cap L^1)$ for a certain $s \geq 2$ and $a \in (0,2)$.

Our aim is to prove a global (in time) existence result.

We will distinguish two cases. In the first one we will suppose low regularity of data, we choose $s = 2$. In the second case we will discuss the model with high regularity of data, so we will choose a suitable large s.

We will see that for $s = 2$ we need more restrictions to the exponent p, to a and to the spatial dimension n. In the second case we shall obtain global existence of small data solutions for all $p > s$ and with a certain $s > 2$ which we specify later.

In both cases the approach to deal (6.1) is similar to the one which we used to treat (5.1). One tool is again Duhamel's principle, as following:

Let us consider the linear problem

$$(u_{lin})_{tt} - \Delta(u_{lin}) - \Delta(u_{lin})_t = 0,\ (u_{lin})(0,x) = \varphi(x),\ (u_{lin})_t(0,x) = \psi(x).$$

According with Chapter 2 one has

$$(u_{lin})(t,x) = G_0(t,x) *_x \varphi(x) + G_1(t,x) *_x \psi(x);$$

being G_0 defined by (3.2) and G_1 defined by (3.3).

Using Duhamel's principle any solution to (5.1) satisfies

$$u(t,x) = G_0(x,t) *_{(x)} \varphi(x) + G_1(x,t) *_{(x)} \psi(x) + \int_0^t G_1(t-s,x) *_{(x)} ||D|^a u(s,x)|^p ds. \quad (6.2)$$

We introduce for all $t \geq 0$ the function space

$$X(t) := C([0,t], H^s) \cap C^1([0,t], H^{s-2})$$

with the norm

$$\begin{aligned} \|w\|_{X(t)} := \sup_{0\leq\tau\leq t} \Big(&(1+\tau)^{\frac{n-2}{4}} \|w\|_{L^2} + (1+\tau)^{\frac{n+2(s-1)}{4}} \||D|^s w\|_{L^2} + (1+\tau)^{\frac{n}{4}} \|w_t\|_{L^2} \\ &+ (1+\tau)^{\frac{n+2(s-2)}{4}} \||D|^{s-2} w_t\|_{L^2} \Big). \end{aligned} \quad (6.3)$$

More precisely, if $n = 2$, then we introduce the norm

$$\begin{aligned} \|w\|_{X(t)} := \sup_{0\leq\tau\leq t} \Big(&(\log(e+t))^{-1} \|w\|_{L^2} + (1+\tau)^{\frac{n+2(s-1)}{4}} \||D|^s w\|_{L^2} + (1+\tau)^{\frac{n}{4}} \|w_t\|_{L^2} \\ &+ (1+\tau)^{\frac{n+2(s-2)}{4}} \||D|^{s-2} w_t\|_{L^2} \Big), \end{aligned}$$

the term $(\log(e+t))^{-1}$ brings no additional difficulties so we will ignore it. It is easy to prove that $\left(X(t), \|\cdot\|_{X(t)}\right)$ is a Banach space. We also define

$$\|w\|_{X_0(t)} := \sup_{0\leq\tau\leq t} \left((1+\tau)^{\frac{n-2}{4}} \|w\|_{L^2} + (1+\tau)^{\frac{n+2(s-1)}{4}} \||D|^s w\|_{L^2} \right). \quad (6.4)$$

Our aim is to prove that for any data (φ, ψ) from the space $A := \left(H^s \cap L^1\right) \times \left(H^{s-2} \cap L^1\right)$ the operator N which is defined for any $w \in X(t)$ by

$$N: \ w \ \rightarrow \ Nw := G_0(x,t) *_{(x)} \varphi(x) + G_1(x,t) *_{(x)} \psi(x) + \int_0^t G_1(t-s,x) *_{(x)} ||D|^a w(s,x)|^p ds$$

satisfies the following estimates:

$$\|Nu\|_{X(t)} \lesssim \|(\varphi,\psi)\|_A + \|u\|^p_{X(t)}, \tag{6.5}$$

$$\|Nu - Nv\|_{X(t)} \lesssim \|u-v\|_{X(t)} \left(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)}\right), \tag{6.6}$$

uniformly with respect to $t \geq 0$. As we explained before, we may conclude that for small initial data the operator N has a unique fixed point which is also the unique solution of (6.1). Finally we notice that due to the definition of $X(t)$ is chosen in an appropriate way to obtain the decay estimates for the solution to the semi-linear problem, too.

As we said the strategy is quite similar to the proof of Theorem 5.2.1. We will use a generalized Gagliardo-Nirenberg inequality which we have already introduced in Theorem 5.1.1.

In the case with low regular data we shall see that the range of admissible p is upper-bounded and this bring more restrictions to the dimension n and to a. To avoid such restrictions we assume more regularity for the data. This regularity is in some sense related to the Gagliardo-Nirenberg inequality. In this way, for large s we just need $p > s$ and no upper-bound appears for p.

6.2 Global existence and decay behaviour

We will distinguish two cases. In the first case we suppose $s = 2$.

Theorem 6.2.1.
Let $A := (H^2 \cap L^1) \times (L^2 \cap L^1)$, $n \geq 2$ *and* $a \in (0,2)$. *Let*

$$p \in \left[2, \frac{n}{n+2(a-2)}\right] \quad \textit{such that} \quad p > \frac{2+n}{n+a-1}. \tag{6.7}$$

Then there exists $\varepsilon > 0$ such that for any $(\varphi, \psi) \in A$ with $\|(\varphi, \psi)\|_A < \varepsilon$ there exists a unique solution $u \in C([0,\infty), H^2) \cap C^1([0,\infty), L^2)$ to (6.1). Moreover, the solution, its first derivative in time and its derivatives in space up to the second order satisfy the following decay estimates:

$$\|u(t,\cdot)\|_{L^2} \lesssim \begin{cases} (1+t)^{-\frac{n-2}{4}} \|(\varphi,\psi)\|_A & \text{if } n \geq 3; \\ \log(e+t)\|(\varphi,\psi)\|_A & \text{if } n = 2; \end{cases}$$
$$\|u_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{4}} \|(\varphi,\psi)\|_A;$$
$$\||D|^2 u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n+2}{4}} \|(\varphi,\psi)\|_A.$$

Remark 9.
Under these assumptions we need

$$2 \leq n \leq 7 \text{ and } a \in \left(0, 2 - \frac{n}{4}\right], \tag{6.8}$$

otherwise the interval $\left[2, \frac{n}{n+2(a-2)}\right]$ is empty. Moreover, under these assumptions we see that condition (6.7) is reasonable, that is, we have $\dfrac{2+n}{n+a-1} < \dfrac{n}{n+2(a-2)}$.

Proof.
As in the previous cases we will prove (6.5) and (6.6). Let us prove the first one. Firstly, we estimate the L^2 norm of Nu. Thanks to Theorems 2.1.4 and 2.1.6 we have that

$$\begin{aligned}\|Nu(t,\cdot)\|_{L^2} &\lesssim (1+t)^{-\frac{n-2}{4}} \|(\varphi,\psi)\|_A + \left\| \int_0^t G_1(t-\tau,x) *_{(x)} ||D|^a u(\tau,x)|^p d\tau \right\|_{L^2} \\ &\lesssim (1+t)^{-\frac{n-2}{4}} \|(\varphi,\psi)\|_A + \int_0^t (1+t-\tau)^{-\frac{n-2}{4}} \|||D|^a u(\tau,x)|^p\|_{L^2\cap L^1} d\tau.\end{aligned}$$

We have $\|||D|^a u(\tau,x)|^p\|_{L^2\cap L^1} \lesssim \||D|^a u\|^p_{L^p} + \||D|^a u\|^p_{L^{2p}}$. To estimate $\||D|^a u\|^p_{L^{kp}}$, $k = 1, 2$, we use (5.6) with $\sigma = 2$, $q = kp$ and $m = 2$. In this way we get

$$\||D|^a u\|_{L^{kp}} \leq \||D|^2 u\|_{L^2}^{\theta_{a,2}(kp,2)} \|u\|_{L^2}^{1-\theta_{a,2}(kp,2)}, \tag{6.9}$$

where $\theta_{a,2}(kp,2) = \dfrac{n}{2}\left(\dfrac{1}{2} - \dfrac{1}{kp} + \dfrac{a}{n}\right) \in \left[\dfrac{a}{n}, 1\right)$ as in Corollary 5.1.3, that is,

$$2 \leq kp < \frac{2n}{n-2(2-a)} \quad \text{or} \quad 2 \leq kp \quad \text{if} \quad \frac{n}{2(2-a)} \leq 1.$$

It means:

- if $n \geq 4$, then we have $\dfrac{n}{2(a-2)} \leq 1$ for no $a \in (0,2)$,
- if $n = 3$, then we have $\dfrac{n}{2(a-2)} \leq 1$ for $a \in \big(0, \frac{1}{2}\big]$,
- if $n = 2$, then we have $\dfrac{n}{2(a-2)} \leq 1$ for $a \in (0,1]$.

Since $\theta_{a,2}(p,2) < \theta_{a,2}(2p,2)$ by using (2.5) and (2.17) with $m = 2$ and $\beta = 0$ we have that

$$\begin{aligned}\| | |D|^a u(\tau,x)|^p \|_{L^2 \cap L^1} &\lesssim \|u\|^p_{X_0(t)} (1+\tau)^{-\frac{n+2}{4}p\theta} (1+\tau)^{-\frac{n-2}{4}p(1-\theta)} \\ &\lesssim \|u\|^p_{X(t)} (1+\tau)^{-\frac{kp(n+a-1)-n}{2k}},\end{aligned}$$

where $\theta = \theta_{a,2}(p,2)$. By using (3.15) we can conclude that

$$\begin{aligned}\int_0^t (1+t-\tau)^{-\frac{n-2}{4}} \| | |D|^a u(\tau,x)|^p \|_{L^2 \cap L^1} d\tau \leq & \|u\|^p_{X(t)} \Big((1+t)^{-\frac{n-2}{4}} \int_0^{t/2} (1+\tau)^{-\frac{p(n+a-1)-n}{2}} d\tau \\ & + \int_{t/2}^t (1+t-\tau)^{-\frac{n-2}{4}} (1+\tau)^{-\frac{p(n+a-1)-n}{2}} d\tau \Big).\end{aligned}$$

Let us examine the integral $\int_0^{t/2}$. Due to (6.7) we have $\frac{p(n+a-1)-n}{2} > 1$. Hence, the term $(1+\tau)^{-\frac{p(n+a-1)-n}{2}}$ is integrable. For the integral $\int_{t/2}^t$ one can repeat the reasoning for estimating Nu in the semi-linear visco-elastic model with source term. In this way we may conclude

$$(1+t)^{\frac{n-2}{4}} \|Nu(t,\cdot)\|_{L^2} \lesssim \|(\varphi,\psi)\|_A + \|u\|^p_{X_0(t)} \lesssim \|(\varphi,\psi)\|_A + \|u\|^p_{X(t)}. \tag{6.10}$$

Let us estimate the term $\partial_t^j |D|^k Nu$, where $(j.k) \in \{(1,0);(0,2)\}$. We will divide the interval $[0,t]$ in two sub-intervals $[0,t/2]$ and $[t/2,t]$. In the first one we need $L^2 \cap L^1 - L^2$ estimates and in the second one we need $L^2 - L^2$ estimates. We will use inequalities (2.5) and (2.17) for the first interval and inequalities (2.9) and (2.19) for the second interval. Then in a very similar way to the previous case

$$\begin{aligned}\|\partial_t^j |D|^k Nu(t,\cdot)\|_{L^2} \lesssim & (1+t)^{-\frac{n+2(j+k-1)}{4}} \|(\varphi,\psi)\|_A \\ & + \|u\|^p_{X(t)} \Big(\int_0^{t/2} (1+t-\tau)^{-\frac{n+2(j+k-1)}{4}} (1+\tau)^{-\frac{p(n+a-1)-n}{2}} d\tau \\ & + \int_{t/2}^t (1+t-\tau)^{-\frac{j+k-1}{2}} (1+\tau)^{-\frac{2p(n+a-1)-n}{4}} d\tau \Big).\end{aligned}$$

For the integral $\int_0^{t/2}$ we repeat our approach from the previous case. Due to (6.7) we have

$$\int_0^{t/2} (1+t-\tau)^{-\frac{n+2(j+k-1)}{4}} (1+\tau)^{-\frac{p(n+a-1)-n}{2}} d\tau \lesssim (1+t)^{-\frac{n+2(j+k-1)}{4}}.$$

For the integral $\int_{t/2}^{t}$ we estimate as follows:

$$\int_{t/2}^{t}(1+t-\tau)^{-\frac{j+k-1}{2}}(1+\tau)^{-\frac{2p(n+a-1)-n}{4}}d\tau$$
$$\lesssim (1+t)^{-\frac{n+2(j+k-1)}{4}}\int_{t/2}^{t}(1+t-\tau)^{-\frac{j+k-1}{2}}(1+\tau)^{-\frac{2p(n+a-1)-2n-2(j+k-1)}{4}}d\tau$$
$$\lesssim (1+t)^{-\frac{n+2(j+k-1)}{4}}.$$

Summarizing we proved

$$(1+t)^{\frac{n}{4}}\|\partial_t Nu(t,\cdot)\|_{L^2}\lesssim \|(\varphi,\psi)\|_A+\|u\|^p_{X_0(t)}\lesssim \|(\varphi,\psi)\|_A+\|u\|^p_{X(t)}; \tag{6.11}$$
$$(1+t)^{\frac{n+2}{4}}\||D|^2 Nu(t,\cdot)\|_{L^2}\lesssim \|(\varphi,\psi)\|_A+\|u\|^p_{X_0(t)}\lesssim \|(\varphi,\psi)\|_A+\|u\|^p_{X(t)}. \tag{6.12}$$

Combining (6.10), (6.11) and (6.12) we completed the proof of (6.5).

Let us prove (6.6). We will use the same approach as before. We know that

$$\|Nu-Nv\|_{X(t)}\lesssim \left\|\int_0^t G_1(t-s,x)*_{(x)}\left(||D|^a u(\tau,x)|^p-||D|^a v(s,x)|^p\right)d\tau\right\|_{X(t)}.$$

Hence, after using (2.5) we have

$$\|(Nu-Nv)(t,\cdot)\|_{L^2}\lesssim \int_0^t(1+t-\tau)^{-\frac{n-2}{4}}\|||D|^a u(\tau,x)|^p-||D|^a v(\tau,x)|^p\|_{L^2\cap L^1}d\tau.$$

Due to (3.16) and by Hölder's inequality it holds

$$\|||D|^a u(\tau,x)|^p-||D|^a v(\tau,x)|^p\|_{L^k}$$
$$\lesssim \||D|^a u(\tau,\cdot)-|D|^a v(s,\cdot)\|_{L^{kp}}\left(\||D|^a u(\tau,\cdot)\|^{p-1}_{L^{kp}}+\||D|^a v(\tau,\cdot)\|^{p-1}_{L^{kp}}\right),$$

where $k=1,2$. Applying generalized Gagliardo-Nirenberg inequality gives

$$\||D|^a u(\tau,\cdot)-|D|^a v(\tau,\cdot)\|_{L^{kp}}\lesssim \||D|^2 u-|D|^2 v\|^{\theta}_{L^2}\|u-v\|^{1-\theta}_{L^2},$$
$$\||D|^a u(\tau,\cdot)\|^{p-1}_{L^{kp}}\lesssim \||D|^2 u\|^{\theta(p-1)}_{L^2}\|u\|^{(1-\theta)(p-1)}_{L^2},$$

where $\theta=\theta_{a,2}(kp,2)=\frac{n}{2}\left(\frac{1}{2}-\frac{1}{kp}+\frac{a}{n}\right)$ and $k=1,2$. So we have the estimates

$$\||D|^a u(\tau,\cdot)-|D|^a v(\tau,\cdot)\|_{L^{kp}}\lesssim \|u-v\|_{X(\tau)}(1+\tau)^{-\frac{1}{2kp}(kp(n-1+a)-n)}, \tag{6.13}$$
$$\||D|^a u(\tau,\cdot)\|^{p-1}_{L^{kp}}\lesssim \|u\|_{X(\tau)}(1+\tau)^{-\frac{p-1}{2kp}(kp(n-1+a)-n)}, \tag{6.14}$$

and, finally,

$$\||D|^a u(\tau,x)|^p - ||D|^a v(\tau,x)|^p\|_{L^k} \lesssim \|u-v\|_{X(t)} \left(\|u\|^{p-1}_{X(t)} - \|v\|^{p-1}_{X(t)}\right)(1+\tau)^{-\frac{1}{2k}(kp(n-1+a)-n)}. \tag{6.15}$$

For this reason, in a very similar way to the previous cases we may conclude the following estimates:

$$(1+\tau)^{\frac{n-2}{4}}\|(Nu-Nv)(t,\cdot)\|_{L^2} \lesssim \|u-v\|_{X(t)}\left(\|u\|^{p-1}_{X(t)} + \|v\|^{p}_{X(t)}\right), \tag{6.16}$$

$$(1+\tau)^{\frac{n}{4}}\|\partial_t(Nu-Nv)(t,\cdot)\|_{L^2} \lesssim \|u-v\|_{X(t)}\left(\|u\|^{p-1}_{X(t)} + \|v\|^{p}_{X(t)}\right), \tag{6.17}$$

$$(1+\tau)^{\frac{n+2}{4}}\||D|^2(Nu-Nv)(t,\cdot)\|_{L^2} \lesssim \|u-v\|_{X(t)}\left(\|u\|^{p-1}_{X(t)} + \|v\|^{p}_{X(t)}\right). \tag{6.18}$$

Combining (6.16), (6.17) and (6.18) we proved (6.6) and this completes the proof. □

Remark 10.
In Theorem 6.2.1 we have some restrictions to the exponent a and to the dimension n. If we want to avoid such restrictions we should require more regularity for the data. This will be done in the next statement.

Theorem 6.2.2.
Let $A := (H^{s+2}\cap L^1)\times(H^s\cap L^1)$. Let us consider the Cauchy problem (6.1) with initial data $(\varphi,\psi)\in A$ and $a\in(0,2)$. Let

$$s > \max\left\{a+\frac{n}{2}, \frac{n+8}{n+2a}\right\}. \tag{6.19}$$

Then there exists $\varepsilon>0$ such that for any $(\varphi,\psi)\in A$ with $\|(\varphi,\psi)\|_A<\varepsilon$ and for any $p>s$ there exists a unique solution $u\in C([0,\infty),H^s)\cap C^1([0,\infty),H^{s-2})$ to (6.1). Moreover u, u_t, $|D|^s u$ and $|D|^{s-2}u_t$ satisfy the following decay estimates:

$$\|u(t,\cdot)\|_{L^2} \lesssim \begin{cases} (1+t)^{-\frac{n-2}{4}}\|(\varphi,\psi)\|_A & \text{if } n\geq 3;\\ \log(e+t)\|(\varphi,\psi)\|_A & \text{if } n=2;\end{cases}$$

$$\|u_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{4}}\|(\varphi,\psi)\|_A$$

$$\||D|^{s+2}u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n+2(s+1)}{4}}\|(\varphi,\psi)\|_A;$$

$$\||D|^{s}u_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n+2s}{4}}\|(\varphi,\psi)\|_A.$$

Proof.
Our goal is to prove (6.5) and (6.6). Let us prove the first one. First step is the estimate of the L^2-norm of Nu. Thanks to Theorems 2.1.4 and 2.1.6 we have

$$\begin{aligned}\|Nu(t,\cdot)\|_{L^2} &\lesssim (1+t)^{-\frac{n-2}{4}}\|(\varphi,\psi)\|_A + \left\|\int_0^t G_1(t-\tau,x) *_{(x)} \||D|^a u(\tau,x)|^p d\tau\right\|_{L^2}\\ &\lesssim (1+t)^{-\frac{n-2}{4}}\|(\varphi,\psi)\|_A + \int_0^t (1+t-\tau)^{-\frac{n-2}{4}}\||\,|D|^a u(\tau,x)|^p\|_{L^2\cap L^1} d\tau.\end{aligned}$$

We have $\||\,|D|^a u(\tau,x)|^p\|_{L^2\cap L^1} \lesssim \||D|^a u\|^p_{L^p} + \||D|^a u\|^p_{L^{2p}}$. To estimate $\||D|^a u\|^p_{L^{kp}}$, $k=1,2$, we use (5.6) with $\sigma = s$, $q = kp$ and $m = 2$. Then we get

$$\||D|^a u\|_{L^{kp}} \leq \||D|^s u\|_{L^2}^{\theta_{a,s}(kp,2)}\|u\|_{L^2}^{1-\theta_{a,s}(kp,2)}, \tag{6.20}$$

where $\theta_{a,s}(kp,2) = \dfrac{n}{s}\left(\dfrac{1}{2}-\dfrac{1}{kp}+\dfrac{a}{n}\right) \in \left[\dfrac{a}{n},1\right)$ as in Corollary 5.1.3, that is,

$$2 \leq kp < \frac{2n}{n-2(s-a)} \quad \text{or} \quad 2 \leq kp \quad \text{if} \quad \frac{n}{2(s-a)} \leq 1,$$

but due to (6.19) we have $\dfrac{n}{2(s-a)} < 1$ and $p > 2$. By using (2.5) and (2.17) with $m = 2$ and $\beta = s-2$ we have that

$$\begin{aligned}\||\,|D|^a u(\tau,\cdot)|^p\|_{L^{kp}} &\lesssim \|u\|^p_{X_0(\tau)}(1+\tau)^{-\frac{n+2(s-1)}{4}p\theta}(1+\tau)^{-\frac{n-2}{4}p(1-\theta)}\\ &\lesssim \|u\|^p_{X(\tau)}(1+\tau)^{-\frac{kp(n+a-1)-n}{2k}},\end{aligned}$$

where $\theta = \theta_{a,s}(kp,2)$. Since $\theta_{a,s}(p,2) < \theta_{a,s}(2p,2)$ and by using (3.15) we can conclude that

$$\begin{aligned}\int_0^t (1+t-\tau)^{-\frac{n-2}{4}}\||\,|D|^a u(\tau,x)|^p\|_{L^2\cap L^1} d\tau \leq &\|u\|^p_{X(t)}\Big((1+t)^{-\frac{n-2}{4}}\int_0^{t/2}(1+\tau)^{-\frac{p(n+a-1)-n}{2}}d\tau\\ &+\int_{t/2}^t (1+t-\tau)^{-\frac{n-2}{4}}(1+\tau)^{-\frac{p(n+a-1)-n}{2}}d\tau\Big).\end{aligned}$$

Let us examine the integral $\int_0^{t/2}$. Due to (6.19) for any $p > s$ we have $\frac{p(n+a-1)-n}{2} > 1$. Consequently, the term $(1+\tau)^{-\frac{p(n+a-1)-n}{2}}$ is integrable. For the integral $\int_{t/2}^t$ one can repeat the reasoning for estimating Nu in the semi-linear visco-elastic model with source term, and then we may conclude

$$(1+t)^{\frac{n-2}{4}}\|Nu(t,\cdot)\|_{L^2} \lesssim \|(\varphi,\psi)\|_A + \|u\|^p_{X(t)}. \tag{6.21}$$

Let us estimate the term $\partial_t Nu$. We will divide the interval $[0,t]$ in two sub-intervals $[0,t/2]$ and $[t/2,t]$. In the first one we need $L^2\cap L^1-L^2$ estimates and in the second one we need L^2-L^2 estimates. We will use inequalities (2.5) and (2.17) in the first interval and inequalities (2.9) and (2.19) in the second part. Then in a very similar way to the previous case we obtain

$$\begin{aligned}\|\partial_t Nu(t,\cdot)\|_{L^2} \lesssim &(1+t)^{-\frac{n}{4}}\|(\varphi,\psi)\|_A\\ &+\|u\|^p_{X(t)}\Big(\int_0^{t/2}(1+t-\tau)^{-\frac{n}{4}}(1+\tau)^{-\frac{p(n+a-1)-n}{2}}d\tau\\ &+\int_{t/2}^t(1+\tau)^{-\frac{2p(n+a-1)-n}{4}}d\tau\Big).\end{aligned}$$

For the integral $\int_0^{t/2}$ we repeat the approach of previous case. Due to (6.19) for all $p>s$ we have

$$\int_0^{t/2}(1+t-\tau)^{-\frac{n}{4}}(1+\tau)^{-\frac{p(n+a-1)-n}{2}}d\tau\lesssim(1+t)^{-\frac{n}{4}}.$$

For the integral $\int_{t/2}^t$ we estimate as follows:

$$\begin{aligned}\int_{t/2}^t(1+\tau)^{-\frac{2p(n+a-1)-n}{4}}d\tau&\lesssim(1+t)^{-\frac{n}{4}}\int_{t/2}^t(1+\tau)^{-\frac{2p(n+a-1)-2n}{4}}d\tau\\ &\lesssim(1+t)^{-\frac{n}{4}}.\end{aligned}$$

Summarizing we proved that

$$(1+t)^{\frac{n}{4}}\|\partial_t Nu(t,\cdot)\|_{L^2}\lesssim\|(\varphi,\psi)\|_A+\|u\|^p_{X(t)}. \tag{6.22}$$

Now let us estimate the term $\||D|^sNu\|_{L^2}$. We will proceed as above. We will divide the interval $[0,t]$ in two sub-intervals $[0,t/2]$ and $[t/2,t]$. We will use $L^2\cap L^1-L^2$ estimates for the first sub-interval and L^2-L^2 estimates for the second one. So by using (2.18) and (2.20) we have

$$\begin{aligned}\||D|^sNu(t,\cdot)\|_{L^2}\lesssim&(1+t)^{-\frac{n+2(s-1)}{4}}\|(\varphi,\psi)\|_A\\ &+\int_0^{t/2}(1+t-\tau)^{-\frac{n+2(s-1)}{4}}\|||D|^au(\tau,x)|^p\|_{H^{s-2}\cap L^1}d\tau\\ &+\int_{t/2}^t(1+t-\tau)^{-\frac{s-1}{2}}\|||D|^au(\tau,x)|^p\|_{H^{s-2}}d\tau\\ \lesssim&(1+t)^{-\frac{n+2(s-1)}{4}}\|(\varphi,\psi)\|_A\\ &+\int_0^{t/2}(1+t-\tau)^{-\frac{n+2(s-1)}{4}}\left(\|||D|^au(\tau,x)|^p\|_{L^2\cap L^1}+\|||D|^au(\tau,x)|^p\|_{\dot H^{s-2}}\right)d\tau\\ &+\int_{t/2}^t(1+t-\tau)^{-\frac{s-1}{2}}\left(\|||D|^au(\tau,x)|^p\|_{L^2}+\|||D|^au(\tau,x)|^p\|_{\dot H^{s-2}}\right)d\tau.\end{aligned}$$

The integrals with $\||D|^a u|^p\|_{L^2\cap L^1}$ or $\||D|^a u|^p\|_{L^2}$ will be handled as before if we apply (6.19). Let us estimate the term $\||D|^a u|^p\|_{\dot H^{s-a}}$. Then by Proposition 5.1.2 it follows

$$\begin{aligned}\||D|^a u(\tau,\cdot)|^p\|_{\dot H^{s-2}} &\lesssim \||D|^a u\|_{\dot H^{s-2}}\||D|^a u\|_{L^\infty}^{p-1} \lesssim \||D|^a u\|_{\dot H^{s-2}}\||D|^a u\|_{H^{s_0}}^{p-1}\\ &\lesssim \||D|^{s+a-2}u\|_{L^2}\left(\||D|^a u\|_{L^2}+\||D|^{s_0+a}u\|_{L^2}\right)^{p-1},\end{aligned}$$

where $s-a>s_0>\dfrac{n}{2}$. We shall estimate the terms $\||D|^{s+a-2}u\|_{L^2}, \||D|^a u\|_{L^2}$ and $\||D|^{s_0+a}u\|_{L^2}$. After using Gagliardo-Nirenberg inequality it follows

$$\begin{aligned}\||D|^{s+a-2}u(\tau,\cdot)\|_{L^2} &\lesssim (1+\tau)^{-\frac{n+2s+2a-6}{4}}\|u\|_{X_0(\tau)},\\ \||D|^a u(\tau,\cdot)\|_{L^2} &\lesssim (1+\tau)^{-\frac{n+2a-2}{4}}\|u\|_{X_0(\tau)},\\ \||D|^{s_0+a}u(\tau,\cdot)\|_{L^2} &\lesssim (1+\tau)^{-\frac{n+2a+2s_0-2}{4}}\|u\|_{X_0(\tau)}.\end{aligned}$$

Hence,

$$\||D|^a u(\tau,\cdot)|^p\|_{\dot H^{s-2}} \lesssim (1+\tau)^{-\frac{p(n+2a-2)+2(s-2)}{4}}\|u\|^p_{X(\tau)}. \tag{6.23}$$

If $s>\frac{8}{n+2a}$, then $\frac{p(n+2a-2)+2(s-2)}{4}>1$ is valid for any $p>s$. But (6.19) gives $s>\frac{8+n}{n+2a}>\frac{8}{n+2a}$ for all $a\in(0,2)$ and $n\geq 2$. Then due to (6.23), it holds

$$\begin{aligned}&\int_0^{t/2}(1+t-\tau)^{-\frac{n+2(s-1)}{4}}\||D|^a u(\tau,x)|^p\|_{\dot H^{s-2}}d\tau\\ &\qquad\lesssim (1+t)^{-\frac{n+2(s-1)}{4}}\|u\|^p_{X(t)}\int_0^{t/2}(1+\tau)^{-\frac{p(n+2a-2)+2(s-2)}{4}}d\tau\\ &\qquad\lesssim (1+t)^{-\frac{n+2(s-1)}{4}}\|u\|^p_{X(t)}.\end{aligned}$$

In an analogous way we estimate

$$\begin{aligned}&\int_{t/2}^{t}(1+t-\tau)^{-\frac{s-1}{2}}\||D|^a u(\tau,x)|^p\|_{\dot H^{s-2}}d\tau\\ &\qquad\lesssim (1+t)^{-\frac{n+2(s-1)}{4}}\|u\|^p_{X(t)}\int_{t/2}^{t}(1+t-\tau)^{-\frac{s-1}{2}}(1+\tau)^{-\frac{p(n+2a-2)+2(s-2)-n-2(s-1)}{4}}d\tau.\end{aligned}$$

On $[t/2,t]$ we have

$$(1+t-\tau)^{-\frac{s-1}{2}}(1+\tau)^{-\frac{p(n+2a-2)+2(s-2)-n-2(s-1)}{4}} \lesssim (1+t-\tau)^{-\frac{p(n+2a-2)-n+2(s-2)}{4}}.$$

Due to (6.19) we have $\frac{p(n+2a-2)-n+2(s-2)}{4}>1$ for all $p>s$. So, $(1+\tau)^{-\frac{p(n+2a-2)-n+2(s-2)}{4}}$ is integrable. Summarizing we proved that

$$(1+t)^{\frac{n+2(s-1)}{4}}\||D|^s Nu(t,\cdot)\|_{L^2} \lesssim \|(\varphi,\psi)\|_A+\|u\|^p_{X(t)}. \tag{6.24}$$

In the same way one can show that

$$(1+t)^{\frac{n+2(s-2)}{4}}\|\partial_t|D|^{s-2}Nu(t,\cdot)\|_{L^2} \lesssim \|(\varphi,\psi)\|_A+\|u\|^p_{X(t)}. \tag{6.25}$$

Combining (6.21), (6.22), (6.24) and (6.25) the proof of (6.5) is completed.

Finally, let us prove (6.6). We will use the same approach of the previous case. We know that

$$\|Nu - Nv\|_{X(t)} \lesssim \left\| \int_0^t G_1(t-s,x) *_{(x)} \big(||D|^a u(\tau,x)|^p - ||D|^a v(s,x)|^p\big) d\tau \right\|_{X(t)}.$$

Hence, by using (2.5) we have that

$$\|(Nu - Nv)(t,\cdot)\|_{L^2} \lesssim \int_0^t (1+t-\tau)^{-\frac{n-2}{4}} \| ||D|^a u(\tau,x)|^p - ||D|^a v(\tau,x)|^p \|_{L^2 \cap L^1} d\tau.$$

Due to (3.16) and by Hölder's inequality it holds

$$\begin{aligned} &\| ||D|^a u(\tau,x)|^p - ||D|^a v(\tau,x)|^p \|_{L^k} \\ &\qquad \lesssim \| |D|^a u(\tau,\cdot) - |D|^a v(\tau,\cdot) \|_{L^{kp}} \left(\| |D|^a u(\tau,\cdot) \|_{L^{kp}}^{p-1} + \| |D|^a v(\tau,\cdot) \|_{L^{kp}}^{p-1} \right), \end{aligned}$$

where $k = 1,2$. By using Gagliardo-Nirenberg inequality we have that

$$\begin{aligned} \| |D|^a u(\tau,\cdot) - |D|^a v(s,\cdot) \|_{L^{kp}} &\lesssim \| |D|^s u - |D|^s v \|_{L^2}^{\theta} \|u - v\|_{L^2}^{1-\theta}, \\ \| |D|^a u(\tau,\cdot) \|_{L^{kp}}^{p-1} &\lesssim \| |D|^s u \|_{L^2}^{\theta(p-1)} \|u\|_{L^2}^{(1-\theta)(p-1)}, \end{aligned}$$

where $\theta = \theta_{a,s}(kp,2) = \dfrac{n}{s}\left(\dfrac{1}{2} - \dfrac{1}{kp} + \dfrac{a}{n}\right)$ and $k = 1,2$. So we may conclude the estimates

$$\| |D|^a u(\tau,\cdot) - |D|^a v(\tau,\cdot) \|_{L^{kp}} \lesssim \|u - v\|_{X(\tau)} (1+\tau)^{-\frac{1}{2kp}(kp(n+a-1)-n)}, \tag{6.26}$$

$$\|u_t(\tau,\cdot)\|_{L^{kp}}^{p-1} \lesssim \|u\|_{X(\tau)} (1+\tau)^{-\frac{p-1}{2kp}(kp(n-1+a)-n)}, \tag{6.27}$$

and, finally,

$$\| ||D|^a u(\tau,\cdot)|^p - ||D|^a v(\tau,\cdot)|^p \|_{L^k} \lesssim \|u - v\|_{X(t)} \left(\|u\|_{X(t)}^{p-1} + \|v\|_{X(t)}^{p-1} \right) (1+\tau)^{-\frac{1}{2k}(kp(n-1+a)-n)}. \tag{6.28}$$

For this reason, in a very similar way to the previous case we may conclude

$$(1+t)^{\frac{n-2}{4}} \|(Nu - Nv)(t,\cdot)\|_{L^2} \lesssim \|u - v\|_{X(t)} \left(\|u\|_{X(t)}^{p-1} + \|v\|_{X(t)}^{p-1} \right), \tag{6.29}$$

$$(1+t)^{\frac{n}{4}} \|\partial_t (Nu - Nv)(t,\cdot)\|_{L^2} \lesssim \|u - v\|_{X(t)} \left(\|u\|_{X(t)}^{p-1} + \|v\|_{X(t)}^{p-1} \right). \tag{6.30}$$

Now we estimate the term $|D|^s(Nu - Nv)$. We will divide the interval $[0,t]$ in two sub-intervals $[0,t/2]$ and $[t/2,t]$. In the first one we need $L^2 \cap L^1 - L^2$ estimates and in the second one we need $L^2 - L^2$ estimates. By using a similar approach to the proof of (6.5) we have

$$\begin{aligned} &\| |D|^s (Nu - Nv)(t,\cdot) \|_{L^2} \\ &\qquad \lesssim \int_0^{t/2} (1+t-\tau)^{-\frac{n+2(s-1)}{4}} \| ||D|^a u(\tau,x)|^p - ||D|^a v(\tau,x)|^p \|_{H^{s-2} \cap L^1} d\tau \\ &\qquad\qquad + \int_{t/2}^t (1+t-\tau)^{-\frac{s-1}{2}} \| ||D|^a u(\tau,x)|^p - ||D|^a v(\tau,x)|^p \|_{H^{s-2}} d\tau. \end{aligned}$$

We have

$$\begin{aligned}&\||D|^a u(\tau,\cdot)|^p - ||D|^a v(\tau,\cdot)|^p\|_{H^{s-2}\cap L^k}\\&\qquad\lesssim \|||D|^a u(\tau,\cdot)|^p - ||D|^a v(\tau,\cdot)|^p\|_{\dot{H}^{s-2}} + \|||D|^a u(\tau,\cdot)|^p - ||D|^a v(\tau,\cdot)|^p\|_{L^2\cap L^k}\end{aligned}$$

for $k=1,2$. The integrals with $\|||D|^a u(\tau,\cdot)|^p - ||D|^a v(\tau,\cdot)|^p\|_{L^2\cap L^1}$ or $\|||D|^a u(\tau,\cdot)|^p - ||D|^a v(\tau,\cdot)|^p\|_{L^2}$ will be handled as before. Now let us estimate $\|||D|^a u(\tau,\cdot)|^p - ||D|^a v(\tau,\cdot)|^p\|_{\dot{H}^{s-2}}$. By using (3.16) and (5.8) we have that

$$\begin{aligned}\|||D|^a u(\tau,\cdot)|^p - ||D|^a v(\tau,\cdot)|^p\|_{\dot{H}^{s-2}} &\lesssim \||D|^a u - |D|^a v\|_{\dot{H}^{s-2}}\left(\||D|^a u\|^{p-1}_{L^\infty} + \||D|^a v\|^{p-1}_{L^\infty}\right)\\&\lesssim \||D|^a u - |D|^a v\|_{\dot{H}^{s-2}}\left(\||D|^a u\|^{p-1}_{H^{s_0}} + \||D|^a v\|^{p-1}_{H^{s_0}}\right),\end{aligned}$$

where $s-a>s_0>\frac{n}{2}$. By using Gagliardo-Nirenberg inequality in a similar way as for the previous cases gives

$$\||D|^a u(\tau,\cdot) - D|^a v(\tau,\cdot)\|_{\dot{H}^{s-2}} \lesssim (1+\tau)^{-\frac{n+2s+2a-6}{4}}\|u(\tau,\cdot)-v(\tau,\cdot)\|_{X(\tau)}, \tag{6.31}$$

$$\begin{aligned}\||D|^a u(\tau,\cdot)\|^{p-1}_{H^{s_0}} &\lesssim \||D|^a u(\tau,\cdot)\|^{p-1}_{\dot{H}^{s_0}} + \||D|^a u(\tau,\cdot)\|^{p-1}_{L^2}\\&\lesssim (1+\tau)^{-\frac{n+2a-2}{4}(p-1)}\|u\|^{p-1}_{X(\tau)}.\end{aligned} \tag{6.32}$$

Then we may conclude

$$\begin{aligned}&\|||D|^a u(\tau,\cdot)|^p - ||D|^a v(\tau,\cdot)|^p\|_{\dot{H}^{s-2}}\\&\qquad\lesssim (1+\tau)^{-\frac{p(n+2a-2)+2(s-2)}{4}}\|u-v\|_{X(\tau)}\left(\|u\|^{p-1}_{X_0(\tau)} + \|v\|^{p-1}_{X_0(\tau)}\right).\end{aligned} \tag{6.33}$$

By using the same arguments as before we derive

$$(1+t)^{\frac{n+2(s-1)}{4}}\||D|^s(Nu(t,\cdot)-Nv(t,\cdot))\|_{L^2} \lesssim \|u-v\|_{X(t)}\left(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)}\right), \tag{6.34}$$

$$(1+t)^{\frac{n+2(s-2)}{4}}\|\partial_t|D|^{s-2}(Nu(t,\cdot)-Nv(t,\cdot))\|_{L^2} \lesssim \|u-v\|_{X(t)}\left(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)}\right). \tag{6.35}$$

Combining (6.29), (6.30), (6.34) and (6.35) we prove (6.6). In this way the proof is completed. □

Bibliography

[1] M. D'Abbicco, S. Lucente and M. Reissig, *Semi-linear wave equations with effective damping*. Chin. Ann. Math. 34B (2012) 345-380.

[2] M. D'Abbicco and M. Reissig, *Semilinear structural damped waves*. Math. Meth. Appl. Sci. (2013) doi: 10.1002/mma.2913

[3] P. T. Duong, M. Kainane and M. Reissig *Decay estimates for structurally damped σ-evolution models*, submitted for publication.

[4] L. C. Evans, *Partial Differential Equations, Second Edition*, American Mathematical Society, 2010.

[5] A. Friedman, *Partial Differential Equations*, Krieger-New York, 1976.

[6] H. Hajaiej, L. Molinet, T. Ozawa, T. and B. Wang, *Necessary and sufficient conditions for the fractional Gagliardo-Nirenberg inequalities and applications to Navier-Stokes and generalized boson equations*, Harmonic analysis and nonlinear partial differential equations, 159-175, RIMS Kokyuroku Bessatsu, B26, Res.Inst.Math.Sci. (RIMS), Kyoto, (2011).

[7] M. Reissig, *PDE Course for Master Students*, Lecture notes for "Messaggeri della conoscenza program". Dipartimento di Matematica, Università degli studi di Bari, 2013.

[8] T. Runst and W. Sickel, *Sobolev spaces of fractional order, Nemytskij operators and nonlinear partial differential equations*, De Gruyter series in nonlinear analysis and applications, Walter de Gruyter & Co., Berlin, 1996.

[9] Y. Shibata, *On the Rate of Decay of Solutions to Linear Viscoelastic Equation*, Math. Meth. Appl. Sci. 23 (2000) 203-226.