Coriolis force from the Lagrangian

January 26, 2022

Abstract

Development of an idea found in Susskind's book [1], about finding the Coriolis force only from the kinetic energy

1 Sliding frames

First, let us consider two coordinate systems: a fixed one (x, y, z) and a moving one (X, Y, Z). We consider rectilinear motion, along axes x and X (they may be aligned, without loss of generality.) The x coordinate from the fixed one to the moving one is L, and it changes with time, $L = L(t)$. The transformation from the moving frame to the fixed one reads

$$\begin{cases} X &= x - L \\ Y &= y \\ Z &= z. \end{cases} \tag{1}$$

Notice an observer at $X = 0$ sees an object at $x = 0$ as being in the negative direction, $X = -L$ (if L is assumed positive.)

The time derivative of x and X are linked by

$$V_x = v_x - U,$$

while $V_y = v_y$, and $V_z = v_z$. The velocity U is the relative velocity between frames, $U = dL/dt$.

The kinetic energy can be written as

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left[(V_x + U)^2 + V_y^2 + V_z^2\right].$$

Writing the Lagrangian

$$L = T - E$$

(where E is the potential energy, which depends on either set of coordinates), the Euler-Lagrange equation of the X coordinate is

$$\frac{d}{dt}\frac{\partial L}{\partial V_x} = \frac{\partial L}{\partial X}.$$

In this case, only T depends on the velocities, and only E on positions, so

$$\frac{d}{dt}\frac{\partial T}{\partial V_x} = -\frac{\partial E}{\partial X} = F_x,$$

where F_X is the force due to potential E.

The derivative of the kinetic energy is

$$\frac{\partial T}{\partial V_x} = m\left(V_x + U\right),$$

so, finally

$$m\left(A_x + A\right) = F_x \implies mA_x = F_x - mA.$$

In this expression $A = dU/dt$ is the relative acceleration between frames.

The final implication is that if $A = 0$ Newton's equation of motion is the same for both frames (Galilean relativity.) However, if the (X, Y, Z) is an accelerating frame, a fictitious, inertial force appears, which seems to "pull" things to the left (if $A > 0$). A familiar effect in our daily lives as a train car, automobile, airplane, etc., accelerates, and we feel pulled towards the rear.

This inertial forces receive their "fictitious" name because they are not "true" forces: the do not represent a physical interaction. However, they are very real, in the sense that they are felt by the bodies in non-inertial frames. A telling sign of an inertial force is that it is always proportional to the mass of the moving body. A force that would *not* be inertial but nevertheless happens to be proportional to the mass is the gravitational force. This prompted Einstein to investigate whether the gravitational force was in fact some kind of inertial force. We demonstrated this to be the case in his theory of general relativity.

2 Rotating frames

2.1 Coordinates

A *rotation matrix* by an angle θ has the form

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

(For simplicity, we only consider the two-dimensional case.)

Its effect on coordinates is :

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = R(\theta)\begin{pmatrix} x \\ y \end{pmatrix}$$

so that the resulting point at coordinates (x', y') comes from the original one at (x, y) with a rotation around the origin of θ (in the same direction as the sign of θ.)

Consider rotating unit vectors $\hat{\boldsymbol{\imath}}$

$$\hat{\boldsymbol{\imath}}' = \cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}} \tag{2}$$
$$\hat{\boldsymbol{\jmath}}' = -\sin\theta\,\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}}. \tag{3}$$

The inverse transformation is given by

$$\hat{\boldsymbol{\imath}} = \cos\theta\,\hat{\boldsymbol{\imath}}' - \sin\theta\,\hat{\boldsymbol{\jmath}}' \tag{4}$$
$$\hat{\boldsymbol{\jmath}} = \sin\theta\,\hat{\boldsymbol{\imath}}' + \cos\theta\,\hat{\boldsymbol{\jmath}}'. \tag{5}$$

Now, a given point in the plane can be written as

$$\vec{\boldsymbol{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} \tag{6}$$

Writing the rotating vectors as in Eqs. (10),

$$\vec{\boldsymbol{r}} = x\left(\cos\theta\,\hat{\boldsymbol{\imath}}' - \sin\theta\,\hat{\boldsymbol{\jmath}}'\right) + y\left(\sin\theta\,\hat{\boldsymbol{\imath}}' + \cos\theta\,\hat{\boldsymbol{\jmath}}'\right) \tag{7}$$
$$= \left(\cos\theta x + \sin\theta y\right)\hat{\boldsymbol{\imath}}' + \left(-\sin\theta x + \cos\theta y\right)\hat{\boldsymbol{\jmath}}'. \tag{8}$$

The result is clearly another way to write the same vector in terms of the rotating frame:

$$\vec{\boldsymbol{r}} = X\hat{\boldsymbol{\imath}}' + Y\hat{\boldsymbol{\jmath}}', \tag{9}$$

where

$$X = \cos\theta\,x + \sin\theta\,y \tag{10}$$
$$Y = -\sin\theta\,x + \cos\theta\,y. \tag{11}$$

Notice this reads

$$\begin{pmatrix} X \\ Y \end{pmatrix} = R(-\theta)\begin{pmatrix} x \\ y \end{pmatrix}.$$

Indeed, if the rotation is positive ($\theta > 0$), the coordinates in the (X, Y) rotating frame will look as rotated a negative angle. This is the rotation equivalent of $X = x - L$ of Eq. (1)! In [1] this equation has the wrong sign in the angle.

2.2 Velocities

Taking the time derivative of Eq. (6) results in

$$\vec{\boldsymbol{v}} = \dot{x}\,\hat{\boldsymbol{\imath}} + \dot{y}\,\hat{\boldsymbol{\jmath}}$$

(a dot means a time derivative), with the resulting kinetic energy

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right).$$

This is the usual. However, if we suppose the rotation angle changes with time $\theta = \theta(t)$, then taking the time derivative of Eq. (9) results in

$$\vec{\boldsymbol{v}} = \dot{X}\,\hat{\boldsymbol{\imath}}' + \dot{X}\,\hat{\boldsymbol{\jmath}}' + X\,\dot{\hat{\boldsymbol{\imath}}}' + Y\,\dot{\hat{\boldsymbol{\jmath}}}',$$

because the rotation vectors are also changing with time!

It is easy to obtain their time derivatives from Eq. (12):

$$\dot{\hat{\boldsymbol{\imath}}}' = \dot{\theta}\left(-\sin\theta\,\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}}\right) = \dot{\theta}\hat{\boldsymbol{\jmath}}' \tag{12}$$

$$\dot{\hat{\boldsymbol{\jmath}}}' = \dot{\theta}\left(-\cos\theta\,\hat{\boldsymbol{\imath}} - \sin\theta\,\hat{\boldsymbol{\jmath}}\right) = -\dot{\theta}\hat{\boldsymbol{\imath}}'. \tag{13}$$

Indeed, the time derivative of a vector of fixed length will always be perpendicular to that vector, and it is easy to understand the fact that the change in $\hat{\boldsymbol{\imath}}'$ is related to $\hat{\boldsymbol{\jmath}}$, and vice-versa, with the sign change.

This permits to write the velocity as

$$\vec{\boldsymbol{v}} = \dot{X}\,\hat{\boldsymbol{\imath}}' + \dot{X}\,\hat{\boldsymbol{\jmath}}' + \dot{\theta}\left(X\,\hat{\boldsymbol{\jmath}}' - Y\hat{\boldsymbol{\imath}}'\right).$$

An interesting way to write the last summand is as a vector product:

$$\dot{\theta}\left(X\,\hat{\boldsymbol{\jmath}}' - Y\hat{\boldsymbol{\imath}}'\right) = \begin{vmatrix} \hat{\boldsymbol{\imath}}' & \hat{\boldsymbol{\jmath}}' & \hat{\boldsymbol{k}}' \\ 0 & 0 & \dot{\theta} \\ X & Y & Z \end{vmatrix} = \boldsymbol{\omega} \times \boldsymbol{r},$$

with $\boldsymbol{\omega} := \dot{\theta}\hat{\boldsymbol{k}}'$, the usual angular velocity vector, perpendicular to the plane of the rotation. (Of course, $\hat{\boldsymbol{k}}'$ is actually the same as $\hat{\boldsymbol{k}}$, but the expressions look more elegant with it.)

Therefore, we may write

$$\vec{\boldsymbol{v}} = \vec{\boldsymbol{v}}' + \boldsymbol{\omega} \times \boldsymbol{r},$$

which shows that the velocity from the fixed frame is the one measured from the rotating frame, $\vec{\boldsymbol{v}}' = \dot{X}\,\hat{\boldsymbol{\imath}}' + \dot{X}\,\hat{\boldsymbol{\jmath}}'$, plus a term stemming from rotation.

Another expression that is convenient for operation is

$$\vec{\boldsymbol{v}} = \left(\dot{X} - \dot{\theta}Y\right)\hat{\boldsymbol{\imath}}' + \left(\dot{Y} + \dot{\theta}X\right)\hat{\boldsymbol{\jmath}}' \tag{14}$$

2.3 Lagrangian

Since the unitary vectors are mutually perpendicular, it is easy to obtain the kinetic energy in terms of rotating frame quantities:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left[\left(\dot{X} - \dot{\theta}Y\right)^2 + \left(\dot{Y} + \dot{\theta}X\right)^2\right].$$

In the Euler-Lagrange equation of the X coordinate is,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{X}} = \frac{\partial L}{\partial X}$$

where $L = T - E$, but now T depends on velocities *and* positions. We can expand the T as

$$T = \underbrace{\frac{1}{2}m\left(\dot{X}^2 + \dot{Y}^2\right)}_{T'} + \underbrace{\frac{1}{2}m\dot{\theta}^2\left(X^2 + Y^2\right)}_{E'} + \underbrace{m\dot{\theta}\left(-\dot{X}Y + \dot{Y}X\right)}_{E_C}.$$

We can identify three terms. The first, T', is the usual kinetic energy, but built from the velocity components in the rotating frame. The second looks very much like a central potential:

$$E' = \frac{1}{2} m\dot{\theta}^2 \left(X^2 + Y^2\right) = \frac{1}{2} m\dot{\theta}^2 r^2.$$

The reader may recognize it as a centrifugal potential, as will become clearer soon.

The final part is the Coriolis contribution, and mixes position and velocities:

$$E_\mathrm{C} = m\dot{\theta}\left(-\dot{X}Y + \dot{Y}X\right) = m\dot{\theta}\begin{vmatrix} X & Y \\ \dot{X} & \dot{Y}\end{vmatrix}$$

If we suppose E still depends only on positions,

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{X}} = \frac{\partial T}{\partial X} + F_X,$$

where $F_X = -\partial E/\partial X$. We can calculate

$$\frac{\partial T}{\partial \dot{X}} = m\left(\dot{X} - \dot{\theta}Y\right) \tag{15}$$

$$\frac{\partial T}{\partial X} = m\dot{\theta}\left(\dot{Y} + \dot{\theta}X\right). \tag{16}$$

If we assume $\dot{\theta}$ is constant (i.e. constant angular velocity), we arrive at

$$m\left(\ddot{X} - \dot{\theta}\dot{Y}\right) = m\dot{\theta}\left(\dot{Y} + \dot{\theta}X\right) + F_X,$$

or

$$m\ddot{X} = F_X + m\dot{\theta}^2 X + 2m\dot{\theta}\dot{Y} \tag{17}$$

$$m\ddot{Y} = F_Y + m\dot{\theta}^2 Y - 2m\dot{\theta}\dot{X}, \tag{18}$$

where we have added the equation for Y, that can be obtained in the same way.

Two inertial terms arise: the first has the shape of a central force, directed outwards from the origin, of strength $m\dot{\theta}^2 r$. This we recognize as the centrifugal force, which acts like a repulsion from the axis of rotation. Notice $\dot{\theta}r = v_\mathrm{rot}$, the rotation velocity, so we can write this force as mv_rot^2/r, a familiar expression.

The other term is the Coriolis force. It has a strange shape: it depends on the velocities, to begin with. It is also always perpendicular to the velocity in the rotating frame, since we can write

$$\vec{\boldsymbol{F}}_\mathrm{C} = 2m\dot{\theta}\left(\dot{X}\hat{\boldsymbol{\imath}}' + \dot{Y}\hat{\boldsymbol{\jmath}}'\right) = -2m\begin{vmatrix} \hat{\boldsymbol{\imath}}' & \hat{\boldsymbol{\jmath}}' & \hat{\boldsymbol{k}}' \\ 0 & 0 & \dot{\theta} \\ \dot{X} & \dot{Y} & \dot{Z}\end{vmatrix} = -2m\,\boldsymbol{\omega}\times\boldsymbol{v}'$$

References

[1] L. Susskind and G. Hrabovsky. *The Theoretical Minimum: What You Need to Know to Start Doing Physics.* The Theoretical Minimum. Basic Books, 2014.