Curvature equations for statistical structures

Daniel Fox

Escuela Técnica Superior de Arquitectura
Universidad Politécnica de Madrid
daniel.fox@upm.es

GRIEG meets Chopin
Warsaw meeting on geometric methods in science

Curvature conditions for a metric

Kulkarni-Nomizu product: $(h \owedge h)_{ijkl} = 2h_{k[i}h_{j]l}$

- Constant sectional curvature: $Riem = -\frac{\kappa}{n(n-1)}(h \owedge h)$.
- Einstein equation (Einstein tensor, $G = Ric - \frac{1}{2}s_h h$):

$$\underbrace{Ric - \frac{1}{n}sh = 0}_{\text{Geometric}} \iff \underbrace{G + \frac{n-2}{2n}\kappa h = 0}_{\text{Physical}}$$

- Constant scalar curvature: $s = \mathrm{tr}_h Ric = \kappa$ is constant.

Beltrami Theorem

A metric has constant sectional curvature $\iff$ it is projectively flat.

Goal: study similar hierarchies

- for statistical structures and, more generally,
- coupling a metric to a tensor with prescribed symmetries and satisfying certain auxiliary PDEs.

Statistical structures

Statistical structure: (∇, h) such that $\nabla_{[i}h_{j]k} = 0$.

- statistical $\iff$ trilinear form $\mathcal{L}_{ijk} = \nabla_i h_{jk}$ symmetric.
- Special if $\nabla \det h = 0 \iff \mathcal{L}_{ijk} = \nabla_i h_{jk}$ trace-free.
- $\bar{\nabla} = \nabla + \mathcal{L}_{ij}{}^k$ generates with h conjugate special statistical structure $(\bar{\nabla}, h)$ having trilinear form $\bar{\mathcal{L}}_{ijk} = -\mathcal{L}_{ijk}$.
- Self-conjugate $\iff$ ∇ is Levi-Civita of h.

Terminology "statistical" comes from a geometric approach to parametric statistics due to Amari-Chentsov-Lauritzen-others but for a geometer the most natural appearance of such structures is:

A cooriented nondegenerate hypersurface in flat affine space acquires a pair of conjugate special statistical structures, one of which is projectively flat.

"Conformal" statistical structures

$([\nabla],[h])$ is an AH (affine hypersurface) structure if for each $\nabla \in [\nabla]$ and each $h \in [h]$ there is a one-form γ_i such that $\nabla_{[i}h_{j]k} = 2\gamma_{[i}h_{j]k}$.

Given a projective structure $[\nabla]$ and a conformal structure $[h]$ on M^n there is a unique $\nabla \in [\nabla]$ that is aligned with respect to $[h]$, meaning:

$$h^{pq}\nabla_i h_{pq} = nh^{pq}\nabla_p h_{qi} \qquad \text{(alignment condition).}$$

Equivalently $H^{ij} = |\det h|^{1/n}h^{ij}$ is divergence free: $\nabla_p H^{ip} = 0$.

Identify $([\nabla],[h])$ with $(\nabla,[h])$, where $\nabla \in [\nabla]$ is aligned representative.

Locally statistical structure: $([\nabla],[h])$ such that every $p \in M$ contained in an open $U \subset M$ on which there are $\nabla \in [\nabla]$ (not necessarily aligned) and $h \in [h]$ such that (∇, h) is a statistical structure on U.

Lemma: $([\nabla],[h])$ is locally statistical $\iff$ $([\nabla],[h])$ is AH.

AH structures vs. statistical structures

$F_{ij} = -d\gamma_{ij}$, where $\gamma_i = \frac{1}{2n} h^{pq}\nabla_i h_{pq}$, does not depend on h.

- AH structure closed if $F \equiv 0$.
- AH structure exact if there is $h \in [h]$ for which $\gamma \equiv 0$.

The aligned representative of the AH structure $([\nabla], [h])$ generated by the statistical structure (∇, h) is $\tilde{\nabla} = \nabla + 2\sigma_{(i}\delta_{j)}{}^{k}$, where $\sigma_i = \frac{1}{n+2} h^{pq}\nabla_i h_{pq}$. In particular, the following are equivalent:

- ∇ is the aligned representative of $([\nabla], [h])$.
- The statistical structure (∇, h) is special.
- The AH structure $([\nabla], [h])$ is exact.

A closed AH structure is locally statistical for the aligned $\nabla \in [\nabla]$.

Conjugacy and construction of AH structures

AH structure $([\nabla], [h])$ and $h \in [h]$ with Faraday primitive γ_i:

- Trilinear form $\mathcal{L}_{ijk} = \nabla_i h_{jk} - 2\gamma_i h_{jk}$ symmetric, trace-free.
- Cubic torsion $\mathcal{L}_{ij}{}^{k} = h^{kp}\mathcal{L}_{ijp}$independent of $h \in [h]$.
- $\bar{\nabla} = \nabla + \mathcal{L}_{ij}{}^{k}$ generates with $[h]$ the conjugate AH structure, for which $\bar{\nabla}$ is aligned and having cubic torsion $\bar{\mathcal{L}}_{ij}{}^{k} = -\mathcal{L}_{ij}{}^{k}$.

- Self-conjugate AH structures = Weyl structures.

Metric h, one-form γ_i, symmetric trace-free $\mathcal{L}_{ijk} = \mathcal{L}_{(ijk)}$, $t \in \mathbb{R} \implies$

$$\nabla = D + t\mathcal{L}_{ijp}h^{pk} - 2\gamma_{(i}\delta_{j)}{}^{k} + h_{ij}h^{kp}\gamma_p \quad (D = \text{Levi-Civita of } h)$$

generates with $[h]$ an AH structure having aligned representative ∇, cubic torsion $-2t\mathcal{L}_{ij}{}^{k}$, and Faraday primitive γ_i.

Conjugacy corresponds with $t \leftrightarrow -t$.

Affine hypersurfaces

$\mathcal{A}^{n+1}$ = flat affine space with parallel volume Ψ.

For locally convex $M^n \subset \mathcal{A}^{n+1}$, affine normal at p is the tangent to the curve formed by barycenters of slices of M by parallel translates of T_pM.

A different definition is needed in nonconvex case.

Cooriented nondegenerate hypersurface in $(\mathcal{A}^{n+1}, \Psi)$ acquires a pair of conjugate special statistical structures (∇, h) and $(\bar{\nabla}, h)$.

- Second fundamental form + coorientation $\implies [h]$.
- vol_h = volume induced by Ψ selects equiaffine metric $h \in [h]$.
- Affine normal induces an affine connection ∇.
- Codazzi equations $\implies \nabla_{[i}h_{j]k} = 0$.
- Flat projective structure $[\bar{\nabla}]$ via pullback by the conormal Gauss map $M \to \mathbb{P}^+(\mathcal{A}^*)$ generates exact AH conjugate to $([\nabla], [h])$.

Curvature of AH and statistical structures

Curvature of AH $([\nabla],[h])$ means curvature of aligned $\nabla \in [\nabla]$.

Curvature of statistical (∇, h) means curvature of ∇.

$R_{ijkl} = R_{ijk}{}^{p}h_{pl}$ curvature of ∇. $\bar{R}_{ijkl} = \bar{R}_{ijk}{}^{p}h_{pl}$ curvature of $\bar{\nabla}$.

Conjugate projectively flat exact $\iff$ locally an affine hypersurface.

- flat statistical structure = Kähler affine (in sense of Cheng-Yau). Often called Hessian, but better to use Hessian for when $h = \nabla dF$ for a global potential F, so Kähler affine structures are locally Hessian.
- Locally Kähler affine $\iff$ projectively flat AH.
- Flat special Hessian = solution of WDVV or associativity equations.

Curvature of special statistical structures

Codazzi operator: $\mathcal{C}(\mathcal{L}) = 2D_{[i}\mathcal{L}_{j]kl}$, Divergence: $\delta(\mathcal{L}) = D_p\mathcal{L}_{ij}{}^p$

$$-2R_{ij(kl)} = 2\nabla_{[i}\nabla_{j]}h_{kl} = 2\nabla_{[i}\mathcal{L}_{j]kl} = 2D_{[i}\mathcal{L}_{j]kl} = \mathcal{C}(\mathcal{L})_{ijkl},$$
$$\implies \bar{R}_{ijkl} = R_{ijkl} + 2\nabla_{[i}\mathcal{L}_{j]kl} = -R_{ijlk},$$
$$\implies R_{ip}{}^p{}_j = \rho(\bar{R})_{ij} \implies \mathrm{tr}_h\,\rho(R) = \mathrm{tr}_h\,\rho(\bar{R}).$$

Lemma: $\bar{R} - R = \mathcal{C}(\mathcal{L})$ and $\rho(\bar{R}) - \rho(R) = \delta(\mathcal{L})$.

- Curvature is self-conjugate $\iff$ $\mathcal{L}$ is Codazzi.
- Ricci curvature is self-conjugate $\iff$ $\mathcal{L}$ is divergence free.

Curvature tensor of h: $Riem = R + \frac{1}{4}\mathcal{L} \owedge \mathcal{L} + \frac{1}{2}\mathcal{C}(\mathcal{L})$,

Ricci curvature of h: $Ric = \rho(R) + \frac{1}{4}\rho(\mathcal{L} \owedge \mathcal{L}) + \frac{1}{2}\delta(\mathcal{L})$.

Curvature equations for special statistical structures

Special statistical has constant curvature if $R = \alpha h \owedge h$ for $\alpha \in \mathbb{R}$.

Constant curvature $\iff$ projectively flat and self-conjugate curvature.

Key point is $R = \alpha h \owedge h \implies \bar{R} = \alpha h \owedge h \implies \mathcal{L}$ Codazzi.

"Natural" notion of Einstein

A special statistical structure is Einstein if:

- (Naive Einstein) $\rho(R)$ and $\rho(\bar{R})$ are multiples of h.
- (Conservation) Scalar curvature $s = \operatorname{tr}_h \rho(R) = \operatorname{tr}_h \rho(\bar{R})$ is constant.

- Recovers usual Einstein equations if $\nabla = D$.
- If Weyl curvature is self-conjugate, naive Einstein $\implies$ conservation.
- If $\mathcal{L}$ is Codazzi, constancy of s follows from naive Einstein condition.

More restrictive notion

Einstein + self-conjugate curvature.

Example: naive Einstein not Einstein

There exist naive Einstein structures that are not Einstein.

- $h = \sum_{i=1}^{3} dx^i \otimes dx^i$ Euclidean with Levi-Civita connection D.
- $dx^{ijk} = \sum_{\sigma \in S_3} dx^{\sigma(i)} \otimes dx^{\sigma(j)} \otimes dx^{\sigma(k)}$.

$$L = (x_1 + x_3)dx^{112} + (x_1 - x_3)dx^{123} - (x_1 + x_3)dx^{233}$$

- L is trace-free and divergence-free.
- $\nabla = D + L^\sharp$ where $h(L^\sharp(u, v), w) = L(u, v, w)$.
- (∇, h) is naive Einstein special statistical but not Einstein.
- Scalar curvature is a multiple of $x_1^2 + x_3^2$.

Generalizing constant scalar curvature

AH structure $(\nabla, [h])$ on M^n. For $h \in [h]$ and $s = h^{ij}Ric(\nabla)_{ij}$, the one-forms $ds_i + 2s\gamma_i$ and $h^{pq}\nabla_p d\gamma_{qi}$ rescale under change of h, and

$$\begin{aligned}\mathcal{L}^{abc}R_{i(abc)} &= \tfrac{n-2}{n}\left(ds_i + 2s\gamma_i + nh^{pq}\nabla_p d\gamma_{qi}\right) \\ &\quad - h_{ia}h^{pq}\left(\bar{\nabla}_p \bar{S}_q{}^a + \nabla_p S_q{}^a\right) - 2\gamma^p(S_{ip} + \bar{S}_{ip}) \\ &= \tfrac{1}{2}\delta(\mathcal{T})_i + \tfrac{2-n}{2n}\mathcal{L}_i{}^{ab}\delta(\mathcal{L})_{ab} - \tfrac{1}{n}\mathcal{L}_i{}^{ab}(\bar{S}_{ab} - S_{ab}),\end{aligned}$$

where $S_{ij} = R_{(ij)} - \frac{s}{n}h_{ij}$ and $\mathcal{T}_{ij} = \mathcal{L}_{ip}{}^q\mathcal{L}_{jq}{}^p - \frac{1}{2}|\mathcal{L}|^2 g_{ij}$.

- Proof: Adaptation of the usual argument showing the Einstein tensor $R_{ij} - \frac{1}{2}sh_{ij}$ is divergence free by tracing $2\nabla_{[i}R_{j]k} = \nabla_p R_{ijk}{}^p$.
- If $n > 2$: self-conjugate curvature and vanishing of $S_{ij} = \bar{S}_{ij} \implies$

$$(\star) \qquad 0 = ds_i + 2s\gamma_i + nh^{pq}\nabla_p d\gamma_{qi}.$$

- D. Calderbank: $(\star)$ as definition of Einstein for 2-D Weyl structures.

Einstein AH structures

An AH structure $(\nabla, [h])$ with conjugate $(\bar{\nabla}, [h])$ is:

- naive Einstein if for any $h \in [h]$, $R_{(ij)} = \frac{s}{n} h_{ij} = \bar{R}_{(ij)}$.
- Einstein if it is naive Einstein and for every $h \in [h]$ with associated one-form γ_i and scalar curvature s, there holds

$$(\star) \qquad 0 = ds_i + 2s\gamma_i + nh^{pq}\nabla_p d\gamma_{qi}.$$

- Recovers usual notion of Einstein-Weyl.
- By definition, $(\nabla, [h])$ is Einstein if and only if $(\bar{\nabla}, [h])$ is Einstein.
- When $n > 2$, self-conjugate curvature + naive Einstein $\implies$ Einstein.

Aside: Bach tensor

Bach tensor $\mathcal{O}_{ij}$ of Weyl structure $([\nabla], [h])$ is trace-free symmetric tensor

$$\mathcal{O}_{ij} = \nabla^p W_{pij} + W^{pq} W_{pijq} = \tfrac{1}{3-n} \nabla^p \nabla^q W_{pijq} + W^{pq} W_{pijq}.$$

When $n = 4$:

- $\mathcal{O}_{ij}$ is divergence free, arises as first variation of $\int_M |W|^2$.
- $\mathcal{O}_{ij} \equiv 0$ for closed Einstein Weyl and ASD conformal structures.

An AH structure is normal if it has self-conjugate Weyl curvature and self-conjugate generalized Cotton tensor (definition omitted).

There is a Bach tensor for AH structures. When $n = 4$ and the curvature is normal it has the properties:

- Symmetric, divergence-free, and self-conjugate.
- For a closed Einstein AH structure with normal curvature, $\mathcal{O}_{ij} \equiv 0$.
- For a projectively flat AH structure with normal curvature, $\mathcal{O}_{ij} \equiv 0$.

Einstein field equations

Data: metric h on n-manifold with some auxiliary fields Ψ.

- Einstein tensor $G = Ric - \frac{1}{2}s_h h$ is divergence-free.
- Einstein field equations with cosmological constant Λ and divergence-free symmetric stress-energy tensor $\mathcal{T}$:

$$G + \Lambda h = \mathcal{T}.$$

  $\mathcal{T}$ determined by some auxiliary fields Ψ.

- Equivalently $Ric - \mathcal{T} = \kappa h$, where

$$\kappa = s - \operatorname{tr}_h \mathcal{T} = \frac{2(n-1)}{n}s - n\Lambda$$

  is constant provided $n > 2$.

- Geometric Einstein equations correspond with vacuum case $\mathcal{T} \equiv 0$.

Example: Einstein-Maxwell equations

- Electromagnetic 2-form F.
- Quadratic expressions: $(F \circ F)_{ij} = F_i{}^p F_{jp}$. $|F|^2 = h^{ik}h^{jl}F_{ij}F_{kl}$.
- Einstein-Maxwell equations:

$$dF = 0 = d \star F, \qquad Ric - F \circ F = \tfrac{1}{n}(s - |F|^2)h.$$

Equivalently:

$$dF = 0 = d \star F, \quad G + \Lambda h = \mathcal{T}, \quad \Lambda = \tfrac{2-n}{2n}\left(s - \tfrac{n-4}{2(n-2)}|F|^2\right).$$

$\mathcal{T} = F \circ F - \frac{1}{4}|F|^2 h$ is stress-energy tensor.

- $\delta(G) = 0$ and $\delta(\mathcal{T}) = 0 \implies$ constancy of Λ.

Is there a projective flatness condition that traces to give EM equations?

Coupled equations for p-forms

For a p-form $F_{ija_1\dots a_{p-2}} = F_{ijA}$:

$$(F\cdot F)_{ijkl} = \tfrac{2}{3}(F_{k[i}{}^{A}F_{j]lA} - F_{ij}{}^{A}F_{klA}), \qquad \rho(F\cdot F)_{ij} = F_{i}{}^{qA}F_{jqA}.$$

Stress-energy tensor: $\mathcal{T} = \rho(F\cdot F) - \frac{1}{2p}|F|^2 h.$

$$h(X, \delta(\mathcal{T})) = h(\iota(X)F, \delta(F)) - \tfrac{1}{p}h(\iota(X)dF, F) \quad \text{(Baird '08)}.$$

Coupled projective flatness:

$$dF = 0 = d\star F, \quad Riem - \epsilon(F\cdot F) = -\tfrac{\kappa}{n(n-1)}h \owedge h, \quad \epsilon \in \{\pm 1\}$$

Coupled Einstein equations:

$$dF = 0 = d\star F, \qquad G + \Lambda h = \epsilon\mathcal{T}, \iff Ric - \epsilon\,\rho(F\cdot F) = \tfrac{\kappa}{n}h,$$

$dF = 0 = d\star F$ and constancy of $\Lambda = \frac{n-2}{2n}\left(s - \epsilon\frac{n-2p}{p(n-2)}|F|^2\right)$.

Einstein-Maxwell couplings: examples

(h, J) a Kähler structure with Kähler form $\omega(\cdot\,,\cdot\,) = h(J\cdot\,,\cdot\,)$.

- (E. Flaherty '78, C. Lebrun '10) If $n = 4$ and (h, J) has constant scalar curvature, taking the primitive part of the Ricci form,

$$F = \omega + \tfrac{1}{2}(\rho - \tfrac{1}{4}s\omega), \qquad \rho(\cdot\,,\cdot\,) = Ric(J\cdot\,,\cdot\,),$$

  (h, F) solves the Einstein-Maxwell equations.

- (h, J) has constant holomorphic sectional curvature 4κ if and only if

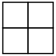
$$Riem - 3\kappa(\omega\cdot\omega) = -\kappa(h \owedge h) \qquad (\kappa = s/24).$$

  If $\kappa > 0$, $F = \sqrt{3\kappa}\omega$, (h, F) is coupled projectively flat with $\epsilon = 1$.

- If $\kappa < 0$, $F = \sqrt{-3\kappa}\omega$, (h, F) is coupled projectively flat with $\epsilon = -1$.

Perhaps interesting question: are there any other solutions?

Auxiliary field data

- $O(n)$-irreducible module $\mathbb{W}$ determines module of trace-free tensors with prescribed symmetries.
- Example:  $\leftrightarrow$ Weyl curvature tensors.
- Symmetric $O(n)$-module map $\phi : \mathbb{W} \times \mathbb{W} \to \mathcal{MC}$.
- Normalization: $\operatorname{tr}_h \rho(\phi(\omega, \omega)) = |\omega|^2$.
- Stress-energy tensor: $\mathcal{T}(\omega) = \rho(\phi(\omega, \omega)) + \frac{1}{2r}|\omega|^2 h$.
- Generalized gradients $\mathcal{A}_1, \dots, \mathcal{A}_N$, symbols $\sigma_{\mathcal{A}_i}(X)(\omega)$.

$$\underbrace{D\omega = \mathop{\Large +}\limits_{i} \mathcal{A}_i(\omega)}_{\text{Definition}}, \quad \underbrace{h(X, \delta(\mathcal{T})) = \sum_{i=1}^{N} \beta_i h(\sigma_{\mathcal{A}_i}(X)(\omega), \mathcal{A}_i(\omega))}_{\text{Compatibility relation}}$$

With minor modifications can include spinors.

General scheme

Coupled projective flatness:

$$\mathcal{A}_i(\omega) = 0, 1 \leqslant i \leqslant N, \quad Riem - \epsilon\phi(\omega,\omega) = -\tfrac{\kappa}{n(n-1)} h \owedge h, \quad \epsilon \in \{\pm 1\}.$$

Coupled Einstein equations:

$$\mathcal{A}_i(\omega) = 0, 1 \leqslant i \leqslant N, \quad G + \Lambda h = \epsilon \mathcal{T} \iff Ric - \epsilon\, \rho(\phi(\omega,\omega)) = \tfrac{\kappa}{n} h.$$

Constant generalized scalar curvature:

$\mathcal{A}_i(\omega) = 0, 1 \leqslant i \leqslant N$, and constancy of $\Lambda = \frac{n-2}{2n}\left(s + \epsilon \frac{n+2r}{r(n-2)}|\omega|^2\right)$.

The existence and behavior of solutions depends strongly on the sign ϵ.

Example: $(3p-1)$-dimensional supergravity

The conditions $\mathcal{A}_i(\omega) = 0$ can be relaxed.

- Data: pseudo-Riemannian (M^n, h), p-form F.
- Suppose $n+1 = 3p$ (e.g. $(p, n) = (2,5), (3,8), (4,11)$, etc.)
- $dF = 0$ and $d \star F = cF \wedge F$, constant $c \implies \delta\mathcal{T} = 0$.

- Coupled projective flatness:

$$dF = 0, \quad d \star F = cF \wedge F, \quad Riem - \epsilon(F \cdot F) = -\tfrac{\kappa}{n(n-1)} h \owedge h.$$

- Variant of supergravity equations:

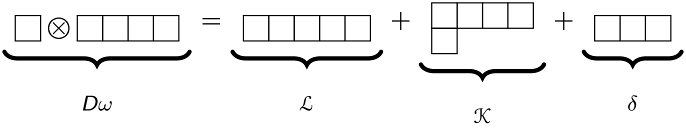
$$\begin{aligned} dF &= 0, \qquad d \star F = cF \wedge F, \\ G + \Lambda h &= \epsilon\mathcal{T} \iff Ric - \epsilon\,\rho(F \cdot F) = \Lambda h. \end{aligned}$$

(For specific c generalizes usual 11-D supergravity equations.)

- $dF = 0$, $d \star F = cF \wedge F$, constancy of $\Lambda = \frac{n-2}{2n}\left(s - \epsilon\frac{n-2p}{p(n-2)}|F|^2\right)$.

Generalized gradients for trace-free Codazzi tensors

Data: Trace-free, symmetric $\omega_{i_1 \dots i_k} = \omega_{(i_1 \dots i_k)}$.



$\mathcal{K}(\omega)$, $\mathcal{L}(\omega)$ trace-free parts of $\mathcal{C}(\omega)_{ijA} = D_{[i}\omega_{j]A}$ and $D_{(i_1}\omega_{i_2 \dots i_{k+1})}$.

Divergence: $\delta(\omega)_{i_1 \dots i_{k-1}} = D^p \omega_{p i_1 \dots i_{k-1}}$.

Conformal Codazzi operator:

$$\mathcal{K}(\omega)_{ij i_1 \dots i_{k-1}} = \mathcal{C}(\omega)_{ij i_1 \dots i_{k-1}} - \frac{1}{n+k-3} \sum_{s=1}^{k-1} h_{i_s [i} \delta(\omega)_{j] i_1 \dots \hat{i}_s \dots i_{k-1}}.$$

Conformal Killing operator:

$$\mathcal{L}(\omega)_{i_1 \dots i_{k+1}} = D_{(i_1}\omega_{i_2 \dots i_{k+1})} - \frac{k}{n+2(k-1)} h_{(i_1 i_2} \delta(\omega)_{i_3 \dots i_{k+1})}.$$

Stress-energy tensors for Codazzi tensors

Data: Trace-free, symmetric $\omega_{i_1 \dots i_k} = \omega_{(i_1 \dots i_k)}$.

- Generalized Kulkarni-Nomizu product and its Ricci trace:

$$(\omega \owedge \omega)_{ijkl} = 2\omega_{k[i}{}^{A}\omega_{j]lA}, \quad \rho(\omega \owedge \omega)_{ij} = (\omega \owedge \omega)_{pij}{}^{p} = \omega_{i}{}^{A}\omega_{jA}.$$

- When $\operatorname{tr}_h \omega = 0$, $\mathcal{C}(\omega) = 0 \implies \delta(\omega) = 0$.

$$\mathcal{T}^{Cod}(\omega) = \rho(\omega \owedge \omega) - \tfrac{1}{2}|\omega|^2 h, \quad \mathcal{T}^{Kil}(\omega) = \rho(\omega \owedge \omega) + \tfrac{1}{2k}|\omega|^2 h.$$

Divergence identities for stress-energy tensors

$$\begin{aligned} h(X, \delta(\mathcal{T}^{Cod}(\omega))) &= -2h(\omega, \iota(X)\mathcal{C}(\omega)) + h(\iota(X)\omega, \delta(\omega)), \\ &= -2h(\iota(X)\mathcal{K}(\omega), \omega) + \tfrac{n-2}{n+k-3} h(\iota(X)\omega, \delta(\omega)), \end{aligned}$$

$$h(X, \delta(\mathcal{T}^{Kil}(\omega))) = \tfrac{k+1}{k} h(\iota(X)\mathcal{L}(\omega), \omega) + \tfrac{n+2k}{n+2(k-1)} h(\iota(X)\omega, \delta(\omega)).$$

Coupled equations for trace-free Codazzi tensors

Data: Trace-free, symmetric $\omega_{i_1 \dots i_k} = \omega_{(i_1 \dots i_k)}$.

Coupled projective flatness:

$$\mathcal{K}(\omega) = 0, \quad \delta(\omega) = 0, \quad Riem - \epsilon(\omega \owedge \omega) = -\tfrac{\kappa}{n(n-1)}(h \owedge h) \quad \epsilon \in \{\pm 1\}.$$

Coupled Einstein equations:

$$\mathcal{K}(\omega) = 0, \quad \delta(\omega) = 0, \quad G + \Lambda h = \epsilon \mathcal{T}(\omega) \iff Ric - \epsilon\, \rho(\omega \owedge \omega) = \tfrac{\kappa}{n} h.$$

Coupled constraint equations:

$$\mathcal{K}(\omega) = 0, \quad \delta(\omega) = 0, \quad \text{constancy of } \Lambda = \tfrac{n-2}{2n}(s - \epsilon|\omega|^2) = \tfrac{n-2}{2n}\kappa.$$

Hierarchy for special statistical structures

Theorem (Fox): A special statistical structure (∇, h) with trilinear form $\mathcal{L}$ and scalar curvature r has

- constant curvature $\iff$ $(h, \mathcal{L})$ solves coupled projectively flat equations with $c = \frac{1}{4}$ and $\kappa = r$;
- self-conjugate curvature and is Einstein $\iff$ $(h, \mathcal{L})$ solves coupled Einstein equations with $c = \frac{1}{4}$ and $\kappa = r$;
- self-conjugate curvature and constant scalar curvature $\iff$ $(h, \mathcal{L})$ solves coupled constraint equations with $c = \frac{1}{4}$ and $\kappa = r$.

Proof follows from:

- Cubic form is trace-free Codazzi if and only if the special statistical structure has self-conjugate curvature.
- For a special statistical structure with self-conjugate curvature:

$$\mathcal{R}_{ijkl} - \frac{1}{4}(\mathcal{L} \owedge \mathcal{L}) = R_{ijkl}, \quad \mathcal{R}_{ij} - \frac{1}{4}\rho(\mathcal{L} \owedge \mathcal{L})_{ij} = R_{ij}, \quad s - \frac{1}{4}|\mathcal{L}|^2 = r.$$

Examples from submanifold geometry

$k = 2$, coupled projectively flat

If h and Π are the induced metric and second fundamental form of a mean curvature zero nondegenerate hypersurface in a pseudo-Riemannian space form, then (h, Π) is coupled projectively flat.

$k = 3$, coupled projectively flat

If h and Π are the induced metric and twisted second fundamental form of a mean curvature zero nondegenerate Lagrangian immersion in a (para/pseudo)-Kähler manifold of constant (para)-holomorphic sectional curvature, then (h, Π) is coupled projectively flat.

Sign ϵ depends on the geometric data - signature, para/pseudo, etc.

Algebraic solutions on a flat background

$k = 3$, h flat

If h flat, ω_{ijk} means it is the trilinear form of a metrized commutative nonassociative algebra. Unique isomorphism class of coupled projectively flat solutions for each $n \geqslant 2$, while coupled Einstein solutions abound.

Hypersurface $\Sigma \subset \mathbb{S}^n$ is isoparametric if has constant principal curvatures. Σ is a level set of a polynomial $P : \mathbb{R}^n \to \mathbb{R}$ homogeneous of degree $g \in \{1, 2, 3, 6\}$ and solving, for $m_i = m_{i+2}$ (indices m 6),

$$(\star) \qquad |dP|^2 = g^2|x|^{2(g-1)}, \qquad \Delta P = \tfrac{m_2 - m_1}{2} g^2 |x|^{g-2}.$$

Isoparametric polynomials

The trace-free part $\omega_{i_1 \dots i_g}$ of polarization of P solving $(\star)$ and Euclidean h solve coupled Einstein equations but are not coupled projectively flat.

Algebraic solutions on nonflat background

G a connected compact simple Lie group with Killing form $B_G < 0$.

- Biinvariant metric $h = -B_G$ on G.
- $\omega_{i_1 \dots i_k} \in S^k(\mathfrak{g}^*)$ the polarization of a harmonic homogeneous G-invariant polyonomial P of degree $k \geqslant 3$.

The pair (h, ω) solves the coupled Einstein equations on G.

Proof: G invariance decouples the equations.

- G-invariance implies $\omega_{i_1 \dots i_k}$ is parallel, so Codazzi.
- G-invariance implies $\rho(\omega \owedge \omega)$ is G-invariant, so is a multiple of h.
- For same reason h is Einstein in usual sense.

In general these solutions not coupled projectively flat. Can be shown for $G = SU(n), k = 3$.

No other examples on compact Lie groups

Theorem (Follows from results of H. T. Laquer, H. Naitoh)

A compact simple Lie group G admits a unique biinvariant exact AH structure $(D, [-B_G])$ unless G has type A_n, $n \geqslant 2$, in which case there is a unique one-parameter family $(\nabla^t, [-B_G])$ of biinvariant exact AH structures. These are Einstein but not constant curvature.

There is a similar theorem for Riemannian symmetric spaces, with exceptional examples on certain symmetric spaces.

Theorem (Follows from results of H. T. Laquer, H. Naitoh)

Each of the irreducible Riemannian symmetric spaces

$SU(n)/SO(n)$	$(SU(n) \times SU(n))/SU(n)$	$SU(2n)/Sp(n)$	E_6/F_4
$n \geqslant 3$	$n \geqslant 3$	$n \geqslant 3$	
$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$

admits a 1-parameter family of solutions of the $k = 3$ coupled Einstein equations. These solutions are not coupled projectively flat.

Einstein special statistical structures and affine spheres

Affine sphere: A nondegenerate hypersurface $M^n \subset \mathcal{A}^{n+1}$ whose affine normals meet in a point (its center) or are parallel (center at infinity).

For nondegenerate cooriented $M^n \subset \mathcal{A}^{n+1}$ with induced conjugate special statistical structures (∇, h) and $(\bar{\nabla}, h)$ the following are equivalent:

- M is an affine sphere.
- (∇, h) and $(\bar{\nabla}, h)$ have self-conjugate curvature.
- (∇, h) and $(\bar{\nabla}, h)$ are Einstein.

These conditions imply and, if $n > 2$, are implied by

- (∇, h) and $(\bar{\nabla}, h)$ are projectively flat.

Yields lots of solutions of the coupled projectively flat equations.

Existence of affine spheres

M a convex affine sphere with complete equiaffine metric h.

- (Blaschke-Deicke-Calabi) M elliptic $\implies$ M ellipsoid.
- (Jörgens-Calabi-Cheng-Yau) M parabolic $\implies$ M elliptic paraboloid.
- (Calabi) M hyperbolic $\implies$ h has nonpositive Ricci curvature.

Theorem (Cheng-Yau '86): The interior of a nonempty pointed closed convex cone is foliated in a unique way by complete hyperbolic affine spheres asymptotic to its boundary and having center at its vertex.

(Klartag, 2018): over a bounded convex domain Ω there is a (necessarily incomplete) elliptic affine sphere with center at the Santaló point of Ω.

Many examples of nonconvex affine spheres from: conditions on cubic form (Dillen, Z. Hu, H. Li, Vrancken), level sets of real forms of relative invariants of irreducible prehomogeneous vector spaces (Fox), nonassociative algebras (Fox, R. Hildebrand).

Cheng-Yau metric

Corollary: a properly convex flat projective manifold carries a canonical homothety class of metrics that constitute with its projective structure an exact AH structure having constant negative curvature.

Proof (Following J. Loftin, 2001): descend the equiaffine metric of an affine sphere asymptotic to the cone over its universal cover.

- Yields abundance of nontrivial compact Riemannian Einstein AH.
- This is like Kähler-Einstein metric in negative Chern class case.
- Analogues of zero and positive Chern class need to be formulated.

Einstein-Weyl structures on bundles

Theorem (Pedersen-Swann, 1991)

- Kähler-Einstein manifold M with scalar curvature $s \neq 0$.
- $N \to M$ principal S^1-bundle with principal connection whose curvature is a multiple of the Kähler form.

N admits a one-parameter family of Einstein-Weyl structures that are Riemannian if $s > 0$ and Lorentzian if $s < 0$.

- Simplest examples are perturbations of Levi-Civitas of Berger metrics on total space of Hopf fibration $\mathbb{S}^3 \to \mathbb{S}^2$.
- Special case of a more general result yielding Einstein-Weyl structures on total spaces of torus bundles over products of Kähler-Einstein manifolds having nonzero first Chern classes.

Negative Chern class case extends to surfaces with a convex flat real projective structure in place of a Kähler-Einstein metric.

Curved cone picture

Theorem (Fox, 2021)

Given an oriented, compact surface M of genus $g \geqslant 2$ equipped with a properly convex flat projective structure $[\nabla]$ having Cheng-Yau metric g and aligned representative ∇, and an integer k such that $|k| \leqslant 2(g-1) = -\chi(M)$, there are a principal S^1-bundle $\rho : N \to M$ with Euler number $e(N) = k$, a principal connection β, and a torsion-free connection $\hat{D}$ on N that together with the Lorentzian metric $G = \beta \otimes \beta - \rho^*(g)$ satisfy:

- $(\hat{D}, [G])$ is Einstein AH with aligned representative $\hat{D}$.
- $\rho : (N, G) \to (M, -g)$ is a metric submersion with timelike and $\hat{D}$-totally geodesic fibers.
- $\nabla_X Y = T\rho(\hat{D}_{\hat{X}} \hat{Y})$ is aligned representative of $[\nabla]$ ($\hat{X}$ is β-lift).
- If $e(N) = \pm\chi(M)$, (∇, g) is a constant curvature Riemannian metric and its Levi-Civita if and only $(\hat{D}, [G])$ is closed.

(Note (∇, g) is special statistical with negative constant curvature.)

Some relevant references

[1] N. Ay, J. Jost, H. V. Lê, and L. Schwachhöfer, Information geometry, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 64, Springer, Cham, 2017.

[2] P. Baird, Stress-energy tensors and the Lichnerowicz Laplacian, J. Geom. Phys. **58** (2008), no. 10, 1329–1342.

[3] E. Calabi, Complete affine hyperspheres. I, Symposia Mathematica, Vol. X (Convegno di Geometria Differenziale, INDAM, Rome, 1971), Academic Press, London, 1972, pp. 19–38.

[4] D. M. J. Calderbank, Two dimensional Einstein-Weyl structures, Glasg. Math. J. **43** (2001), no. 3, 419–424.

[5] S. Y. Cheng and S. T. Yau, The real Monge-Ampère equation and affine flat structures, Proceedings of the 1980 Beijing Symposium on Differential Geometry and Differential Equations, Vol. 1, 2, 3 (Beijing, 1980) (Beijing), Science Press, 1982, pp. 339–370.

[6] D. J. F. Fox, Conelike radiant structures, arXiv:2106.04270.

[7] ______, Einstein equations for a metric coupled to a trace-free symmetric tensor, arXiv:2105.05514.

Some relevant references

[8] ———, Geometric structures modeled on affine hypersurfaces and generalizations of the Einstein Weyl and affine hypersphere equations, arXiv:0909.1897.

[9] ———, Einstein-like geometric structures on surfaces, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **XII** (2013), no. 5, 499–585.

[10] ———, A Schwarz lemma for Kähler affine metrics and the canonical potential of a convex cone, Ann. Mat. Pura Appl. (4) **194** (2015), no. 1, 1–42.

[11] H. T. Laquer, Invariant affine connections on Lie groups, Trans. Amer. Math. Soc. **331** (1992), no. 2, 541–551.

[12] J. C. Loftin, Affine spheres and Kähler-Einstein metrics, Math. Res. Lett. **9** (2002), no. 4, 425–432.

[13] H. Naitoh, Totally real parallel submanifolds in $P^n(c)$, Tokyo J. Math. **4** (1981), no. 2, 279–306.

[14] H. Pedersen and A. Swann, Riemannian submersions, four-manifolds and Einstein-Weyl geometry, Proc. London Math. Soc. (3) **66** (1993), no. 2, 381–399.