TOPICS IN GEOMETRIC ANALYSIS: MAXIMUM PRINCIPLES, REFINED KATO INEQUALITIES, AND GRADIENT ESTIMATES OF HARMONIC FUNCTIONS AND EIGENFUNCTIONS OF THE LAPLACIAN

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1. Introduction

These are notes for lectures given in the Universidad Complutense de Madrid, January 13, 15, 20, 22, 25, and 27 (2 hours each day) of 2016.

The focus was the use of Bochner formulas and refined Kato inequalities to obtain estimates of the growth of geometric quantities. This was illustrated by deducing a sharp form of the local gradient estimate of Cheng and Yau.

The notes have not been carefully editted and so likely contain typographical errors and minor misstatements. References have been given, but no effort has been made to make them complete.

The notes contain some material that was not presented in the lectures and omit much related material. In particular the use of volume comparison results in the analysis of harmonic and subharmonic functions, althought closely related, is not included.

The choice of the contents of the lectures was modeled to some extent on the early chapters of the book [37] of R. Schoen and S. T. Yau and the book [27] of P. Li, both of which are recommended. For the required geometric background, the book [12] of J. Cheeger and D. Ebin seems to still be the best reference. For more refined aspects the articles [8] of E. Calabi and [11] of J. Cheeger are recommended as starting points. In particular, most the geometric ideas mentioned here appear in some form in [8].

2. Preliminaries

2.1. **Notation.** In these notes all manifolds are smooth and connected and without boundary, unless something else is indicated explicitly. The word **smooth** means infinitely differentiable unless something else is indicated explicitly.

If $E \to M$ is a vector bundle, the space of sections of E over $U \subset M$ is denoted $\Gamma(E; U)$. When $U = M$ there is written simply $\Gamma(E)$.

The dual vector bundle is written E^*. The endomorphism bundle is written $\text{End}(E)$. The $k$fold tensor power of E is written $\otimes^k E$. The kth symmetric power of E is written $S^k(E)$. The kth antisymmetric power of E is written $\Omega^k(E)$.

2.2. **Tensors and abstract index notation.** The *abstract index notation* serves to indicate tensors and their symmetries. They were systematized by R. Penrose (see [33]). Another reference that describes the approach is [42]. Since the abstract index conventions are used frequently in these notes, and sometimes cause confusion for those unaccustomed to them, there are described now.

Indices are formal labels or placeholders and do not indicate a choice of frame, although if one is fixed indices can be interpreted as indicating components with respect to it. An index has both a horizontal position and a vertical position. The vertical position of an index is referred to as *up* position or *down*. Tensors that transform covariantly (sections of tensor powers of the cotangent bundle) are indicated using down indices. Tensors that transform contravariantly (sections of tensor powers of the tangent bundle) are indicated using up indices. For example a vector field can be written as X or as X^i and a one-form as α or as α_i. The two notations X and X^i are exactly synonymous. The index is merely a label that indicates the nature of the object labeled; in X^i it indicates a section of the tangent bundle. Here a virtue of the abstract index notation is already apparent. When one writes X or α, the character of the tensor (covariant or contravariant, how many components, etc.) is not apparent from the notation. However, when one writes X_i and α^j it is apparent that X is a one-form and α is a vector field.

Enclosure of indices in square brackets indicates complete antisymmetrization over the enclosed indices. Enclosure of indices in parentheses indicates complete symmetrization over the enclosed indices. For example, $a^{ij} = a^{(ij)}+a^{[ij]}$ indicates the decomposition of a contravariant two-tensor into its symmetric and skew-symmetric parts. Inclusion of an index between vertical bars $||$ indicates its omission from an indicated symmetrization; for example $2a_{[i|jk|l]} = a_{ijkl} - a_{ljki}$. For another example, the exterior product $\alpha \wedge \beta \in \Gamma(\Omega^{p+q}T^*M)$ of $\alpha \in \Gamma(\Omega^p T^*M)$ and $\beta \in \Omega^q T^*M)$ is defined by

$$(\alpha \wedge \beta)_{i_1 \dots i_{p+q}} = \binom{p+q}{p} \alpha_{[i_1 \dots i_p} \beta_{i_{p+1} \dots i_{p+q}]}. \tag{2.1}$$

(Note that many authors define the exterior product using a different convention for the combinatorial coefficient; when doing Hodge theory the convention given by (2.1) is inconvenient.) The symmetric product $\alpha \odot \beta \in \Gamma(S^{p+q}T^*M)$ of $\alpha \in \Gamma(S^p T^*M)$ and $\beta \in S^q T^*M)$ is defined by

$$(\alpha \odot \beta)_{i_1 \dots i_{p+q}} = \alpha_{(i_1 \dots i_p} \beta_{i_{p+1} \dots i_{p+q})}. \tag{2.2}$$

With the conventions used here, the symmetric product of α and β is by definition the completely symmetric part of their ordinary tensor product. On the other hand, the exterior product of forms is not the completely antisymmetric part of their tensor product; rather it is a combinatorial multiple of this anti-symmetric part. The different conventions for the combinatorial factors in the definitions (2.1) and (2.2) of the exterior and symmetric products reflect the author's habits and nothing more. Every convention is convenient in some contexts and inconvenient in others.

The summation convention is always used in the following form. A label appearing as both an up and a down index indicates the trace pairing, that is contraction of the tensor components indicated by the given labels. For example $X^p \alpha_p$ indicated the evaluation $\alpha(X)$ of the one-form α on the vector field X. The tautological pairing $\Gamma(TM) \times \Gamma(T^*M) \to C^\infty(M)$ can be viewed as a section of $\mathrm{End}(TM)$ (or of $\mathrm{End}(T^*M)$), and is written $\delta_i{}^j$. So $X^p \alpha_p = X^i \delta_i{}^j \alpha_j$. On the other hand, the Kronecker delta δ_{ij} does not have its usual sense in the abstract index notations. Writing δ_{ij} simply indicates a covariant two-tensor, and nothing more.

If a frame or coframe is fixed, then an abstract index representation of a tensor can be interpreted as the components of the tensor with respect to the given frame. However this will not be done much in these notes.

2.3. **Riemannian metrics.** A *pseudo-Riemannian metric* (or, simply, a *metric*) is an everywhere nondegenerate section $g \in \Gamma(S^2(T^*M))$. Since a metric is everywhere nondegenerate, its signature is well defined. A metric is *Riemannian* if it is everywhere positive definite. The terminology is not ideal; a pseudo-Riemannian metric could be Riemannian.

Using partitions of unity it is straightforward to show that every smooth manifold admits a Riemannian metric. The analogous claim for metrics of other signatures is false. For example, a compact orientable surface admits a split signature metric if and only if it is a torus; the null space of the metric is a global line field, and this forces the Euler characteristic to be zero.

If g_{ij} is a metric, the inverse symmetric bivector g^{ij} is defined by $g_{ip}g^{pj} = \delta_j{}^i$. When a metric is fixed, indices are raised and lowered using g_{ij} and g^{ij} consistently with the conventions $X^i = g^{ip}X_p$ and $\alpha_i = g_{ip}\alpha^p$.

If M is oriented, the volume form vol_g determined by the Riemannian metric g is the unique n-form such that $\mathsf{vol}_g(E_1, \dots, E_n) = 1$ for an oriented g-orthonormal frame $\{E_1, \dots, E_n\} \subset \Gamma(TM)$.

2.4. **Tensor norms.** There are at least two standard ways to define norms on a space of tensors. Here the norm $|T|$ of a tensor $T_{j_1\dots j_q}^{i_1\dots i_p} \in \Gamma(\otimes^p TM \otimes \otimes^q T^*M)$ is defined by *complete contraction* with the metric g:

$$|T|^2 = T^{i_1\dots i_p}{}_{j_1\dots j_q}T^{k_1\dots k_p}{}_{l_1\dots l_q}g_{i_1k_1}\dots g_{i_pk_p}g^{j_1l_1}\dots g^{j_ql_q} = T^{i_1\dots i_p}{}_{j_1\dots j_q}T_{i_1\dots i_p}{}^{j_1\dots j_q}. \tag{2.3}$$

The norm on a given tensor bundle given by complete contraction in general differs by a constant factor from the norm *induced by g*. The norm induced by g is defined by choosing an orthonormal frame $\{E_1, \dots, E_n\}$ in TM, and declaring that appropriate linear combinations of tensor products of the elements of the frame, generating the space of tensors in question, constitute an orthonormal basis. For example, the norm $\|\cdot\|$ induced on $\Omega^2(TM)$ is that with respect to which $\{E_i \wedge E_j : 1 \leq i < j \leq n\}$ constitutes an orthonormal basis. For example, if $X, Y \in \Gamma(TM)$ are unitary and orthogonal, then $\|X \wedge Y\|^2 = 1$, whereas, since $(X \wedge Y)^{ij} = 2X^{[i}Y^{j]}$,

$$|X \wedge Y|^2 = 4X^{[i}Y^{j]}X_{[i}Y_{j]} = 4X^iY^jX_{[i}Y_{j]} = 2(|X|^2|Y|^2 - g(X,Y)^2) = 2 = 2\|X \wedge Y\|^2. \tag{2.4}$$

In these notes the norms defined by complete contraction are used always. This is different from the conventions in many textbooks and papers.

2.5. **Connections on vector bundles.** M is a smooth n-dimensional manifold.

A *connection* (or *covariant derivative*) on a smooth vector bundle E over M is a linear map $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$ such that

$$\nabla(fs) = s \otimes df + f\nabla s \tag{2.5}$$

for all $f \in C^\infty(M)$ and $s \in \Gamma(E)$. For $X \in \Gamma(TM)$ one writes $\nabla_X : \Gamma(E) \to \Gamma(E)$ for the linear operator defined by

$$\nabla_X s = \langle \nabla s, X \rangle, \tag{2.6}$$

where $\langle \cdot\, , \cdot \rangle : \Gamma(T^*M) \times \Gamma(TM) \to C^\infty(M)$ is the canonical pairing given by duality.

A connection on E induces a connection on any vector subbundle of any tensor product of any tensor powers of E and its dual. It is customary to denote these induced connections by the same symbol, ∇.

Theorem 2.1. *On any smooth manifold M, any smooth vector bundle, E, admits a connection.*

Proof. It is straightforward to construct a connection on a trivial vector bundle. On a general vector bundle, one patches together connections constructed on an open cover comprising locally trivializations using a subordinate partition of unity. □

The *difference tensor*, $\Pi \in \Gamma(\Omega^2(T^*M) \otimes E)$, of two connections ∇ and $\bar{\nabla}$ on E is defined by

$$\Pi(X, Y) = \bar{\nabla}_X Y - \nabla_X Y, \tag{2.7}$$

for $X, Y \in \Gamma(TM)$. This means that the space of connections on E is an affine space modeled on the vector space $\Gamma(\Omega^2(T^*M) \otimes E)$.

The *curvature* $R \in \Gamma(\Omega^2(T^*M) \otimes \mathrm{End}(E))$ of the connection ∇ is defined by

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \tag{2.8}$$

for $X, Y \in \Gamma(TM)$. Some authors define the curvature to be $-R(X, Y)$.

A connection is **flat** if its curvature vanishes identically.

The covariant derivative is a linear map $\nabla : \Gamma(TM) \to \Gamma(\text{End}(E))$. Each of the spaces $\Gamma(TM)$ and $\Gamma(\text{End})$ is a Lie algebra; $\Gamma(TM)$ is a Lie algebra with the Lie bracket of vector fields, and $\Gamma(\text{End}(E))$ is a Lie algebra with the fiberwise Lie bracket of endomorphisms. The curvature measures the failure of ∇ to be a Lie algebra homomorphism.

The connection ∇ is *flat* if its curvature vanishes identically (one says *vanishes identically* for clarity, to distinguish from the situation of vanishing at a point; one means simply that the section, R, of the vector bundle $\Omega^2(T^*M) \otimes \text{End}(E)$ is the zero section).

2.6. **Affine connections.** A connection ∇ on the tangent bundle TM is called an *affine connection*. The *torsion*, $T \in \Gamma(\Omega^2(T^*M) \otimes TM)$ of an affine connection is defined by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \tag{2.9}$$

for $X, Y \in \Gamma(TM)$. An affine connection is *torsion-free* if its torsion vanishes identically.

Every smooth manifold admits a torsion-free affine connection. Let $\bar{\nabla}$ be any affine connection with torsion $\bar{T}$. Then

$$\nabla_X Y = \bar{\nabla}_X Y - \tfrac{1}{2}\bar{T}(X,Y) \tag{2.10}$$

is a torsion-free affine connection.

The space of torsion-free affine connections is an affine space modeled on the vector space $\Gamma(S^2(T^*M) \otimes TM)$.

The curvature of an affine connection is defined by

$$2\nabla_{[i}\nabla_{j]}X^k = R_{ijp}{}^k X^p. \tag{2.11}$$

If ∇ and $\tilde{\nabla} = \nabla + \Pi_{ij}{}^k$ are two torsion-free affine connections, their curvatures $R_{ijk}{}^l$ and $\tilde{R}_{ijk}{}^l$ and Ricci curvatures are related by

$$\begin{aligned} \tilde{R}_{ijk}{}^l - R_{ijk}{}^l &= 2\nabla_{[i}\Pi_{j]k}{}^l + 2\Pi_{p[i}{}^l\Pi_{j]k}{}^p, \\ \tilde{R}_{ij} - R_{ij} &= \nabla_p \Pi_{ij}{}^p - \nabla_i \Pi_{pj}{}^p + \Pi_{pq}{}^q \Pi_{ij}{}^p - \Pi_{ip}{}^q \Pi_{jq}{}^p. \end{aligned} \tag{2.12}$$

For a tensor $\sigma_{i_1\dots i_k}^{j_1\dots j_l}$,

$$2\nabla_{[p}\nabla_{q]}\sigma_{i_1\dots i_k}^{j_1\dots j_k} = -\sum_{r=1}^{k} R_{pqi_r}{}^a \sigma_{i_1\dots i_{r-1}ai_{r+1}\dots i_k}^{j_1\dots j_k} + \sum_{s=1}^{l} R_{pqb}{}^{j_s}\sigma_{i_1\dots i_k}^{j_1\dots j_{s-1}bj_{s+1}\dots j_k} \tag{2.13}$$

This identity is often called the *Ricci identity*. In what follows, it will be used frequently, without further comment.

The Lie derivative along $X \in \Gamma(TM)$ of $\sigma_{i_1\dots i_k}^{j_1\dots j_l}$ can be expressed in terms of any torsion-free affine connection ∇ by

$$(\mathfrak{L}_X\sigma)_{i_1\dots i_k}^{j_1\dots j_l} = X^p\nabla_p\sigma_{i_1\dots i_k}^{j_1\dots j_l} + \sum_{r=1}^{k} \nabla_{i_r}{}^p \sigma_{i_1\dots i_{r-1}pi_{r+1}\dots i_k}^{j_1\dots j_k} - \sum_{s=1}^{l} \nabla_q X^{j_r}\sigma_{i_1\dots i_k}^{j_1\dots j_{s-1}qj_{s+1}\dots j_k}. \tag{2.14}$$

The curvature of an affine connection satisfies the algebraic and differential Bianchi identities. While these can be stated for affine connections with torsion, here they will be needed only for torsion-free affine connections. In this case they take the forms:

$$\begin{aligned} R_{[ijk]}{}^l &= 0, \\ \nabla_{[i}R_{jk]l}{}^p &= 0. \end{aligned} \tag{2.15}$$

The *Ricci tensor* of an affine connection is defined by $R_{ij} = R_{pij}{}^p$.

By the algebraic Bianchi identity, for a torsion-free affine connection, the trace $R_{ijp}{}^{p}$ is related to the Ricci tensor by

$$-R_{ijp}{}^{p} = R_{jpi}{}^{p} + R_{pij}{}^{p} = 2R_{[ij]}. \tag{2.16}$$

This relation shows that all possible traces of the curvature tensor of a torsion-free affine connection yield linear combinations of the Ricci tensor and its skew-symmetric part.

Lemma 2.2.

1. *The skew-symmetric part of the Ricci tensor of a torsion-free affine connection is an exact two-form.*
2. *The Ricci tensor of a torsion-free affine connection ∇ is symmetric if there exists a non-vanishing ∇-parallel density of nontrivial weight.*

Proof. Since $\det T^{*}M = \Omega^{n}(T^{*}M)$ is a line bundle, it makes sense to write $|\det T^{*}M|$ for its tensor product with the orientation bundle. A λ-density is a section of $|\det T^{*}M|^{\lambda}$. All line bundles associated with the frame bundle by one-dimensional representation of the general linear group have the form $|\det T^{*}M|^{\lambda}$ tensored with some power of the orientation bundle. The exponent λ is the *weight*, and it is nontrivial if $\lambda \neq 0$. Locally a 1-density is just a volume form, but a nonorientable manifold admits global nonvanishing 1-densities even though it admits no global nonvanishing volume forms. Alternatively, a 1-density is a smooth measure.

Let $\mu_{i_1\dots i_n}$ be a nonvanishing volume form defined on an open set U. By the Ricci identity,

$$2\nabla_{[a}\nabla_{b]}\mu_{i_1\dots i_n} = -\sum_{s=1}^{n} R_{abi_s}{}^{p}\mu_{i_1\dots i_{s-1}pi_{s+1}\dots i_n} = -R_{abp}{}^{p} = 2R_{[ab]}. \tag{2.17}$$

If there exists a nonvanishing ∇-parallel density of nontrivial weight, then locally there exists a nonvanishing ∇-parallel volume form, and the preceding identity shows the Ricci tensor is symmetric.

Given a nonvanishing 1-density μ, the covariant derivative $\nabla\mu$ can be regarded as a 1-density valued 1-form, so $\gamma = \mu^{-1}\nabla\mu$ is a 1-form. By the Ricci identity,

$$2R_{[ij]} = -R_{ijp}{}^{p} = 2\mu^{-1}\nabla_{[i}\nabla_{j]}\mu = 2\nabla_{[i}\gamma_{j]} = d\gamma_{ij}, \tag{2.18}$$

showing that the skew-symmetric part of the Ricci tensor of ∇ is exact. In particular, if there exists a nonvanishing ∇-parallel density of nontrivial weight, then there exists a nontrivial ∇-parallel 1-density, and so the Ricci tensor of ∇ is symmetric.

□

2.7. Levi-Civita connections.

Theorem 2.3. *Given a metric $g \in \Gamma(S^2(T^{*}M))$, there is a unique torsion-free affine connection D, the Levi-Civita connection of g, such that $\nabla g = 0$.*

Proof. Let ∇ be a torsion-free affine connection and let $D = \nabla + \Pi$ where $\Pi \in \Gamma(S^2(T^{*}M)\otimes TM)$ is to be determined. Then

$$D_i g_{jk} = \nabla_i g_{jk} - \Pi_{ijk} - \Pi_{ikj}. \tag{2.19}$$

where $\Pi_{ijk} = \Pi_{ij}{}^{p}g_{pk}$. Hence

$$\begin{aligned} D_i g_{jk} &+ D_j g_{ik} - D_k g_{ij} \\ &= \nabla_i g_{jk} + \nabla_j g_{ik} - \nabla_k g_{ij} - (\Pi_{ijk} + \Pi_{ikj} + \Pi_{jik} + \Pi_{jki} - \Pi_{kij} - \Pi_{kji}) \\ &= \nabla_i g_{jk} + \nabla_j g_{ik} - \nabla_k g_{ij} - 2\Pi_{ijk}. \end{aligned} \tag{2.20}$$

Hence, if $D_i g_{jk} = 0$ then necessarily $\Pi_{ijk} = \frac{1}{2}(\nabla_i g_{jk} + \nabla_j g_{ik} - \nabla_k g_{ij})$. On the other hand, if Π_{ijk} is defined in this way, then $\Pi_{ijk} + \Pi_{ikj} = \nabla_i g_{jk}$, so $D_i g_{jk} = 0$. □

Let g be a Riemannian metric. Let $x^1, \dots, x^n$ be coordinates on an open set $U \subset M$. Let ∂ be the flat torsion-free affine connection such that the coordinate vector fields ∂_{x^i} are parallel. Then the Levi-Civita connection D of g is given by

$$D = \partial + \tfrac{1}{2}g^{kp}(\partial_i g_{jp} + \partial_j g_{ip} - \partial_p g_{ij}. \tag{2.21}$$

The *curvature* and *Ricci curvature* of a metric g mean the curvature and Ricci curvature of the associated Levi-Civita connection D.

The *scalar curvature* of the metric g is the trace $R_g = g^{ij}R_{ij}$.

The *sectional curvature* $K(L)$ of a 2-dimensional subspace $L \subset T_pM$ spanned by linearly independent vectors $X, Y \in T_pM$ is

$$K(L) = \tfrac{g(R(X,Y)Y,X)}{|X|^2|Y|^2 - g(X,Y)}. \tag{2.22}$$

An n-dimensional Riemannian manifold has **constant sectional curvature** if $K(L)$ is equal to a constant for all L. This is the case if and only if

$$R_{ijkl} = \tfrac{2R_g}{n(n-1)} g_{k[i}g_{j]l}, \tag{2.23}$$

in which case R_g is necessarily constant, as a consquence of the Bianchi identities.

A Riemannian manifold has positive, negative, nonpositive, nonnegative, etc. sectional curvature if the stated condition holds for all choices of L.

2.8. **Geodesics and distance.** A C^2 curve $\gamma : [a, b] \to M$ is a *geodesic* of the affine connection ∇ if $\nabla_{\dot\gamma}\dot\gamma = 0$. This equation is a shorthand, requiring interpretation. Let T be any vector field agreeing with $\dot\gamma$ along $\gamma([a, b])$; then the equation means $\nabla_T T = 0$. In general, in these notes this sort of fussing will be left to the reader. Written in local coordinate, the equation defining a geodesic is a system of second order ordinary differential equations and so a solution is uniquely determined by an initial position $\gamma(0) = p \in M$ and an initial velocity $\dot\gamma(0) = X \in T_pM$.

For an interval $I \subset \mathbb{R}$, the length of a C^1 curve $\gamma : I \to M$ is defined by

$$\ell(ga) = \int_I |\dot\gamma(t)|\, dt. \tag{2.24}$$

The distance $d(p, q)$ between points $p, q \in M$ is the infimum of the length of piecewise C^1 curves joining p to q.

2.9. **Completeness.** A geodesic of an affine connection is *infinitely extendible* if its maximal domain of definition is $\mathbb{R}$. An affine connection is *complete* if every geodesic is infinitely extendible. When it is necessary to distinguish the completeness of an affine connection from completeness in the metric space, one says that the affine connection is *geodesically complete*.

There are two notions of completeness applicable to a Riemannian metric. On the one hand, its Levi-Civita connection can be complete. On the other hand, the induced distance function can be complete in the sense of metric spaces. The Hopf-Rinow theorem states that these two notions of completeness are equivalent, so that there is no ambiguity in saying that a Riemannian manifold is complete. This has the important consequence that a compact Riemannian manifold is complete.

Theorem 2.4 (Hopf-Rinow theorem)**.** *For a Riemannian manifold (M, g) with distance function d and Levi-Civita connection D, the following statements are equivalent.*

1. *D is geodesically complete.*
2. *The metric space (M, d) is complete.*
3. *For some $p \in M$, $\exp_p$ is defined on all of T_pM.*
4. *For all $p \in M$, $\exp_p$ is defined on all of T_pM.*
5. *A bounded, closed subset of M is compact.*

If there hold these conditions, then any two distinct points in M are joined by a length minimizing geodesic.

By the Hopf-Rinow theorem it makes sense to call a Riemannian manifold complete if its Levi-Civita connection is geodesically complete, or its distance function is metrically complete.

Corollary 2.5. *If a Riemannian manifold is compact, it is complete.*

Note that a torsion-free affine connection on a compact manifold need not be complete. The simplest example is given as follows. Let $\mathbb{H}(\lambda)$ be the quotient of $\mathbb{R}^n \setminus \{0\}$ by the action of the abelian group generated by dilations by a factor of $\lambda > 1$. Topologically, $\mathbb{H}(\lambda)$ is a circle bundle over the $(n-1)$-sphere, so is compact. It is called a Hopf manifold. Since these dilations are affine automorphisms, the standard flat affine connection on $\mathbb{R}^n$ descends to $\mathbb{H}(\lambda)$. However, it is geodesically incomplete, as the image of a line in $\mathbb{R}^n$ passing through the origin is a geodesic that is not infinitely extendible.

A long standing conjecture, attributed to Markus, is that a torsion-free affine connection that preserves a volume form on a compact manifold must be geodesically complete. This is known in some special cases, but is open in general, even for flat affine connections, or Levi-Civita connections of metrics of indefinite signature.

The recent paper [1] shows:

Theorem 2.6 ([1]). *If M is a compact manifold and the closure of the holonomy of the torsion-free affine connection ∇ is compact, then ∇ is geodesically complete.*

Since the holonomy of the Levi-Civita connection of a Riemannian metric is contained in the orthogonal group, which is compact, its closure is necessarily compact, and so this theorem has as a corollary that the Levi-Civita connection of a Riemannian metric on a compact manifold is complete.

3. Integration by parts

3.1. **Volume form of a Riemannian metric.** Let M be an n-dimensional manifold. If g is a Riemannian metric on M then $\det g$ is a 2-density. By definition, its value on a local frame $E_1, \dots, E_n$ is the determinant of the matrix whose components are $g(E_i, E_j)$. Since g is positive definite, this determinant is positive, and it makes sense to speak of the 1-density $(\det g)^{1/2}$. Let $\epsilon^1, \dots, \epsilon^n$ be the local coframe dual to the given frame. Then $(\det(g(E_i, E_j))1/2\epsilon^1 \wedge \dots \wedge \epsilon^n$ is a volume form. If M is oriented, the local frames $E_1, \dots, E_n$ can be chosen so that $\epsilon^1 \wedge \dots \wedge \epsilon^n$ is compatible with the given orientation, and $(\det(g(E_i, E_j))1/2\epsilon^1 \wedge \dots \wedge \epsilon^n$ patch together to give a globally defined volume form vol_g compatible with the given orientation of M. By definition the tensor square of vol_g equals $\det g$, and the 1-density $(\det g)^{1/2}$ is identified with the 1-density $|\mathsf{vol}_g|$. Alternatively, the volume-form vol_g is defined by the property that if $E_1, \dots, E_n$ is any local orthonormal frame compatible with the given orientation of M, then $\mathsf{vol}_g(E_1, \dots, E_n) = 1$.

3.2. **Divergence.** If Ψ is a volume form on M, the *divergence of $X \in \Gamma(TM)$ with respect to Ψ* is the function $\operatorname{div}_\Psi(X)$ defined by

$$d(\iota(X)\Psi) = \mathfrak{L}_X\Psi = \operatorname{div}_\Psi(X)\Psi. \tag{3.1}$$

If ∇ is a torsion-free affine connection on M, there is a one-form γ_i such that $\nabla\Psi = \gamma \otimes \Psi$. By (2.14),

$$\begin{aligned} \operatorname{div}_\Psi(X)\Psi_{i_1\dots i_n} &= (\mathfrak{L}_X\Psi)_{i_1\dots i_n} \\ &= X^p\nabla_p\Psi_{i_1\dots i_n} + \sum_{s=1}^n \nabla_{i_s}X^p\Psi_{i_1\dots i_{s-1}i_{s+1}\dots i_n} \\ &= (\gamma(X) + \nabla_pX^p)\Psi_{i_1\dots i_n}. \end{aligned} \tag{3.2}$$

Hence $\operatorname{div}_\Psi(X) = \nabla_pX^p + \gamma(X)$. Consequently, $\operatorname{div}_\Psi(X) = \nabla_pX^p$ for any torsion-free affine connection that preserves Ψ. In particular, if D is the Levi-Civita connection of a metric on an orientable manifold, then the divergence of X with respect to the volume form determined by the metric is given by D_pX^p.

When working with a fixed volume form Ψ and an affine connection ∇ preserving Ψ, it is convenient to omit from $\operatorname{div}_\Psi(X)$ the subscript indicating the dependence on Ψ, and to write simply $\operatorname{div}(X)$.

By definition

$$\begin{aligned} \operatorname{div}([X,Y])\Psi &= \mathfrak{L}_{[X,Y]}\Psi = \mathfrak{L}_X(\mathfrak{L}_Y\Psi) - \mathfrak{L}_Y(\mathfrak{L}_X\Psi) \\ &= \mathfrak{L}_X(\operatorname{div}(Y)\Psi) - \mathfrak{L}_Y(\operatorname{div}(X)\Psi) \\ &= (\mathfrak{L}_X\operatorname{div}(Y) - \mathfrak{L}_Y\operatorname{div}(X) + \operatorname{div}(Y)\operatorname{div}(X) - \operatorname{div}(Y)\operatorname{div}(X))\Psi \\ &= (\mathfrak{L}_X\operatorname{div}(Y) - \mathfrak{L}_Y\operatorname{div}(X))\Psi. \end{aligned} \tag{3.3}$$

Hence

$$\operatorname{div}([X,Y]) = \mathfrak{L}_X\operatorname{div}(Y) - \mathfrak{L}_Y\operatorname{div}(X). \tag{3.4}$$

The identity (3.4) means that the divergence is a cocycle for the Lie algebra cohomology of the Lie algebra of vector fields with coefficients in $C^\infty(M)$.

Lemma 3.1 (Integration by parts)**.** *Let M be an oriented smooth manifold with (possibly empty) boundary ∂M. Let Ψ be a volume form on M. If $X \in \Gamma(TM)$ has compact support then*

$$\int_M \operatorname{div}_\Psi(X)\,\Psi = \int_{\partial M}\iota(X)\Psi. \tag{3.5}$$

In particular, if $\partial M = \emptyset$, then $\int_M \operatorname{div}_\Psi(X)\,\Psi = 0$.

Proof. By Stokes's theorem,

$$\int_M \operatorname{div}_\Psi(X)\,\Psi = \int_M \mathfrak{L}_X\Psi = \int_M d(\iota(X)\Psi)) = \int_{\partial M}\iota(X)\Psi. \tag{3.6}$$

□

Use of the identity (3.5) is referred to as *integration by parts.*

Remark 3.2. Lemma 3.1 continues to make sense on a nonorientable Riemannian manifold if the Riemannian volume form is replaced by the Riemannian volume density.

Example 3.3. If $\Psi = \mathsf{vol}_g$ is the volume element of a Riemannian metric and $X \in \Gamma(TM)$ has compact support,

$$\begin{aligned} \int_M X^pD_qD^qX_p\,d\mathsf{vol}_g &= \int_M (D_q(X^pD^qX_p) - D^qX^pD_qX_p)\ d\mathsf{vol}_g \\ &= -\int_M |DX|^2 d\,\mathsf{vol}_g, \end{aligned} \tag{3.7}$$

the last equality by (3.6), because $D_q(X^pD^qX_p)$ is a divergence.

3.3. **Laplacian.** On a Riemannian manifold (M, g), the Laplacian Δ is the second order elliptic differential operator on functions defined, for $f \in C^2(M)$, by

$$\Delta f = D^d df_p = g^{ij} D_i df_j = \operatorname{div} \operatorname{grad} f, \tag{3.8}$$

where $\operatorname{grad} f$ is the vector field dual to df via the metric g. When it is necessary to indicate the dependence of Δ on g there is written Δ_g.

Let $x^1, \dots, x^n$ be local coordinates on the open set $U \subset M$ and let ∂ be the flat torsion-free affine connection with respect to which the coordinate vector fields ∂_{x^i} are parallel. Since $(\det g)^{-1}\partial_i \det g = g^{ab}\partial_i g_{ab}$ and $\partial_i g^{ip} = -g^{pq}\partial_p g_{qi}$, by (2.21),

$$\begin{aligned} \partial_i((\det g)^{1/2} g^{ip} df_p) &= (\det g)^{1/2} \left(g^{ip}\partial_i df_p + \tfrac{1}{2} g^{ab}\partial_i g_{ab} g^{ip} df_p + df_p \partial_i g^{ip}\right) \\ &= (\det g)^{1/2} g^{pq} \left(\partial_p df_q + \tfrac{1}{2}\left(\partial_i g_{jp} + \partial_j g_{ip} - \partial_p g_{ij}\right) df_q\right) \\ &= (\det g)^{1/2} g^{pq} D_p df_q = (\det g)^{1/2} \Delta_g f. \end{aligned} \tag{3.9}$$

Hence, in local coordinates the Laplacian is given by

$$\begin{aligned} \Delta f &= (\det g)^{-1/2}\partial_i((\det g)^{1/2} g^{ip} df_p) \\ &= g^{ij}\tfrac{\partial^2 f}{\partial x^i \partial x^j} + (\det g)^{-1/2}\partial_i((\det g)^{1/2} g^{ip})\tfrac{\partial f}{\partial x^i}. \end{aligned} \tag{3.10}$$

For an open set $U \in M$, a function $f \in C^2(U)$ is harmonic if $\Delta f = 0$. It is **subharmonic** (resp. superharmonic) if $\Delta u \geq 0$ (resp $\Delta u \leq 0$).

Remark 3.4. It follows from standard elliptic regularity theory that a C^2 harmonic function is in fact infinitely differentiable.

If $\operatorname{grad} f$ is the vector field such that $\iota(\operatorname{grad} f)g = df$, then

$$\mathfrak{L}_{\operatorname{grad} f}\mathsf{vol}_g = (\operatorname{div} \operatorname{grad} f)\mathsf{vol}_g = (\Delta f)\mathsf{vol}_g, \tag{3.11}$$

so that a function be harmonic means that its gradient flow preserves the Riemannian volume.

From the fact that the Laplacian is a divergence, there follows, by integration by parts:

Lemma 3.5. *On a Riemannian manifold (M, g), a compactly supported harmonic function is constant.*

Proof. Suppose M is orientable and let vol_g be the Riemannian volume form compatible with a fixed orientation. Let $f \in C^2(M)$ be a compactly supported harmonic function. By Lemma 3.1,

$$0 = \int_M f\Delta f\,\mathsf{vol}_g = \int_M f \operatorname{div} \operatorname{grad} f\,\mathsf{vol}_g = \int_M \left(\operatorname{div}(f \operatorname{grad} f) - |df|^2\right)\mathsf{vol}_g = -\int_M |df|^2\,\mathsf{vol}_g \leq 0. \tag{3.12}$$

Hence $df = 0$ and f is constant. If M is not orientable, the pullback of a harmonic function f to the connected oriented double cover of M is harmonic with respect to the pullback metric, so is constant; hence f is constant. □

3.4. **Conformal change of metric.** A metric $\tilde{g}_{ij}$ is **conformal** to a metric g_{ij} if $\tilde{g}_{ij} = f g_{ij}$ for some positive function $f \in C^\infty(M)$. Let D and $\tilde{D}$ be the Levi-Civita connections of g and $\tilde{g}$. It is straightforward to check that

$$\tilde{D} = D + \Pi_{ij}{}^k = \partial + \sigma_i \delta_j{}^k + \sigma_j \delta_i{}^k - \delta_{ij}\sigma^k, \tag{3.13}$$

where $2\sigma = d\log f$ and $\sigma^k = g^{kp}\sigma_p$ (because of the uniqueness of the Levi-Civita connection it suffices to check that the connection $\tilde{D}$ defined by (3.13) is torsion-free and preserves $\tilde{g}_{ij}$).

Because $g^{ij}\Pi_{ij}{}^k = (2-n)\sigma^k$, the Laplacians Δ_g and $\Delta_{\tilde{g}}$ are related by

$$\Delta_{\tilde{g}}\phi = \tilde{g}^{ij} D_i d\phi_j = f^{-1}(g^{ij}\partial_i d\phi_j - g^{ij}\Pi_{ij}{}^k d\phi_k) = f^{-1}(\Delta_g f + (n-2)g^{ij}\sigma_i d\phi_j). \tag{3.14}$$

3.5. **Laplacian on hyperbolic space.** Let M be the upper half-space $\{x \in \mathbb{R}^n : x_n > 0\}$ and let δ_{ij} be the Euclidean metric $dx_1^2 + \cdots + dx_n^2$. For $f = x_n^{-2}$ the metric $g_{ij} = x_n^{-2}\delta_{ij}$ is the hyperbolic metric having constant sectional curvature -1. This can be checked using the expression (3.13) for the difference tensor $D - \partial$ in terms of $\sigma = -d\log x_n$, the expression (2.12) for the difference of the curvatures of D and ∂ in terms of this difference tensor, and the fact that ∂ is flat.

By (3.14), for $\phi = x_n^\alpha$,

$$\Delta_g(x_n^\alpha) = x_n^{-2}\Delta_\delta(x_n^\alpha) - (n-2)x_n^{-1}\langle dx_n, d(x_n^\alpha)\rangle) = \alpha(\alpha - (n-1))x_n^\alpha. \tag{3.15}$$

The minimum value of $\alpha(\alpha - (n-1))$ is assumed when $\alpha = (n-1)/2$, and this minimum value is $-(n-1)^2/4$.

By (3.15), x_n^{n-1} is harmonic, and

$$\Delta_g \log x_n = x_n^{-1}\Delta_g x_n - x_n^{-2}|dx_n|_g^2 = (1-n). \tag{3.16}$$

4. Bochner vanishing theorem and generalizations

In [5], S. Bochner showed a pair of parallel theorems, one treating one-forms and the other treating vector fields. The first Bochner theorem yields as a corollary that the isometry group of a compact Riemannian manifold with negative Ricci curvature is finite. The second Bochner theorem shows that a compact Riemannian manifold with positive Ricci curvature has first Betti number zero. The ideas underlying the proofs apply in more general contexts, and will be important subsequently.

A nice overview of the context of the theorems discussed in this section can be found in the survey paper [3] of P. Berard.

4.1. **Affine Killing fields.** The pullback $\phi^*\nabla$ of a torsion-free affine connection ∇ via a diffeomorphism ϕ is defined by

$$T\Phi(\phi^*(\nabla)_X Y) = \nabla_{T\phi X} T\phi Y. \tag{4.1}$$

Differentiating the pullback of ∇ along the local flow generated by $X \in \Gamma(TM)$ yields

$$(\mathfrak{L}_X\nabla)_{ij}{}^k = \nabla_i\nabla_j X^k + X^p R_{pij}{}^k. \tag{4.2}$$

This is a second order linear differential operator on X. A vector field is an **affine Killing field** if it generates a local flow by automorphisms of ∇. This means exactly that $\mathfrak{L}_X\nabla = 0$.

4.2. **Killing fields.** Fix a Riemannian metric g with Levi-Civita connection D on the n-dimensional manifold M. For a vector field $X \in \Gamma(TM)$, let $X^\flat \in \Gamma(T^*M)$ be the one-form defined by $X^\flat(Y) = g(X,Y)$ for $Y \in \Gamma(TM)$. Note that $X_i^\flat$ and X_i are notational synonyms. The redundant notations are useful for writing something like $dX^\flat$ without ambiguity.

If $X \in \Gamma(TM)$, then, by (2.14),

$$(\mathfrak{L}_X g)_{ij} = X^p D_p g_{ij} + D_i X^p g_{pj} + D_j X^p g_{ip} = D_i X_j + D_j X_i = 2D_{(i}X_{j)}. \tag{4.3}$$

Definition 4.1.

*(1) $X \in \Gamma(TM)$ is a **conformal Killing field** if there is $c \in C^\infty(M)$ such that $\mathfrak{L}_X g = cg$.*
*(2) $X \in \Gamma(TM)$ is a **Killing field** if $\mathfrak{L}_X g = 0$.*

Remark 4.1.

(1) The terminology *Killing* honors Wilhelm Killing.
(2) If X is conformal Killing, tracing $cg_{ij} = (\mathfrak{L}_X g)_{ij} = 2D_{(i}X_{j)}$ yields $nc = \operatorname{div}(X)$. Consequently, if M is compact, then c necessarily has mean zero.
(3) The infinitesimal generator of a one-parameter family of (conformal) isometries is a (conformal) Killing field.

(4) The sets of conformal Killing and Killing vector fields form Lie subalgebras of the Lie algebra of vector fields. These can be identified with the Lie algebras of the groups of conformal isometries and isometries.
(5) The group $PSL(2,\mathbb{C})$ acts by conformal isometries on the round sphere (viewed as $\mathbb{P}^1(\mathbb{C})$ with the Fubini-Study metric). This gives the simplest examples of conformal Killing fields that are not Killing.

Lemma 4.2. *If D is the Levi-Civita connection of the Riemannian metric g on the manifold M then*

$$(\mathfrak{L}_X D)_{ijk} = D_{(i}(\mathfrak{L}_X g)_{j)k} - \tfrac{1}{2}D_k(\mathfrak{L}_X g)_{ij}, \tag{4.4}$$

for all $X \in \Gamma(TM)$. In particular, a Killing field for g is an affine Killing field for D.

Proof. Because $(\mathfrak{L}_X g)_{ij} = 2D_{(i}X_{j)}$,

$$\begin{aligned}
&2D_{(i}(\mathfrak{L}_X g)_{j)k} - D_k(\mathfrak{L}_X g)_{ij}\\
&\quad= D_iD_jX_k + D_iD_kX_j + D_jD_iX_k + D_jD_kX_i - D_kD_iX_j - D_kD_jX_i\\
&\quad= 2D_{[i}D_{k]}X_j + 2D_{[j}D_{k]}X_i + 2D_{(i}D_{j)}X_k\\
&\quad= -R_{ikj}\,pX_p - R_{jki}{}^pX_p + 2D_iD_jX_k - 2D_{[i}D_{j]}X_k\\
&\quad= R_{kij}\,pX_p - R_{jki}\,pX_p + R_{ijk}{}^pX_p + 2D_iD_jX_k\\
&\quad= 2D_iD_jX_k - 2R_{jki}{}^pX_p = 2D_iD_jX_k + 2X^pR_{pijk} = 2(\mathfrak{L}_X D)_{ijk}.
\end{aligned} \tag{4.5}$$

□

Theorem 4.3 (K. Yano [43])**.** *On a compact orientable Riemannian manifold (M,g) with Levi-Civita connection D, a vector field is affine Killing for D if and only if it is Killing for g.*

Proof. By (4.4), for any $X \in \Gamma(TM)$, $2(\mathfrak{L}_X D)_{i(jk)} = D_i(\mathfrak{L}_X g)_{jk} = 2D_iD_{(j}X_{k)}$. Hence

$$\begin{aligned}
\int_M D_{(j}X_{k)}D^{(j}X^{k)}\,\mathsf{vol}_g &= \int_M D_{(j}X_{k)}D^jX^k\,\mathsf{vol}_g\\
&= -\int_M XkD^jD_{(j}Xk)\,\mathsf{vol}_g = -\int_M X^k(\mathfrak{L}_X D)^j{}_{(jk)}\,\mathsf{vol}_g.
\end{aligned} \tag{4.6}$$

If $\mathfrak{L}_X D = 0$ then the right-hand side vanishes, and so $D_{(i}X_{j)}$ must vanish identically too. □

4.3. **Bochner theorems.** Fix a Riemannian metric g with Levi-Civita connection D on the n-dimensional manifold M. Let $X \in \Gamma(TM)$, so that $X^\flat = \iota(X)g \in \Gamma(T^*M)$. Note that $dX^\flat)_{ij} = 2D_{[i}X_{j]}$. In a parallel manner define $\mathcal{L}(X)_{ij} = 2D_{(i}X_{j)}$. Using the decomposition $D_iX_j = D_{(i}X_{j)} + D_{[i}X_{j]}$ in two different ways yields the parallel identities:

$$\begin{aligned}
D^pD_pX_i &= -D^pD_iX_p + 2D^pD_{(p}X_{i)} = -D_iD^pX_p + R^p{}_{ip}{}^qX_q + D^p\mathcal{L}(X)_{pi}\\
&= -d\,\mathrm{div}(X)_i + D^p\mathcal{L}(X)_{pi} - R_{ip}X^p.
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
D^pD_pX_i &= D^pD_iX_p + 2D^pD_{[p}X_{i]} = D_iD^pX_p - R^p{}_{ip}{}^qX_q + D^pdX^\flat_{pi}\\
&= d\,\mathrm{div}(X)_i + D^pdX^\flat_{pi} + R_{ip}X^p,
\end{aligned} \tag{4.8}$$

The structural parallelism between (4.7) and (4.8) is discussed further in section 4.4. These yield the parallel formulas:

$$\begin{aligned}\tfrac{1}{2}\Delta|X|^2 &= D^p(X^q D_p X_q) = X^q D^p D_p X_q + D^p X^q D_p X^q \\ &= -X^p D_p \operatorname{div}(X) + X^p D^q \mathcal{L}(X)_{qp} - X^p X^q R_{pq} + |DX|^2,\end{aligned} \tag{4.9}$$

$$\begin{aligned}\tfrac{1}{2}\Delta|X|^2 &= D^p(X^q D_p X_q) = X^q D^p D_p X_q + D^p X^q D_p X^q \\ &= X^p D_p \operatorname{div}(X) + X^p D^q dX^\flat_{qp} + X^p X^q R_{pq} + |DX|^2,\end{aligned} \tag{4.10}$$

Exploiting these parallel formulas in similar manners yields the following parallel theorems, both due to S. Bochner.

Theorem 4.4 (S. Bochner [5]). *Let (M, g) be an n-dimensional compact Riemannian manifold.*

1. *If the Ricci curvature is nonpositive and negative at some point of M, then there is no nontrivial Killing field. Consequently there are no one-parameter groups of isometries of g and the isometry group of g is a finite group.*
2. *If the Ricci curvature is nonpositive, every Killing field is parallel, and the dimension of the space of Killing fields is no greater than n.*

Theorem 4.5 (S. Bochner [5]). *Let (M, g) be an n-dimensional compact Riemannian manifold.*

1. *If the Ricci curvature is nonnegative, then the first Betti number of M is no greater than n.*
2. *If the Ricci curvature is nonnegative and positive at some point of M, then the first Betti number of M is zero.*

For each of Theorem 4.4 and 4.5 there are given two proofs. The proofs are the same except one invokes the maximum principle and the other uses integration by parts. Generalizations of these results proceed by generalizing one of these two techniques.

Proof of Theorem 4.4. If X is a Killing field, then $\mathcal{L}(X) = 0$ and $\operatorname{div}(X) = 0$ so (4.9) yields

$$\tfrac{1}{2}\Delta|X|^2 = -X^p X^q R_{pq} + |DX|^2 \geq -X^p X^q R_{pq} \geq 0. \tag{4.11}$$

Hence $|X|^2$ is subharmonic. Since M is compact, this means $|X|^2$ is constant. In (4.11), this implies $0 = \frac{1}{2}\Delta|X|^2 \geq |DX|^2 \geq 0$, so $|DX|^2 = 0$ and X is parallel. In particular, if X is nontrivial it is nowhere vanishing. If the Ricci curvature is somewhere negative and X is not identically zero, then (4.11) implies $0 = \frac{1}{2}\Delta|X|^2 \geq -X^p X^q R_{pq} > 0$, a contradiction. In general, the a vector subbundle spanned by parallel vector fields can have dimension no more then n. □

Alternative proof of Theorem 4.4. If X is a Killing field, then $\mathcal{L}(X) = 0$ and $\operatorname{div}(X) = 0$ so (4.9) yields (4.11). Integrating (4.11) over M yields

$$\int_M X^p X^q R_{pq}\, \mathsf{dvol}_g = \int_M |DX|^2\, \mathsf{dvol}_g. \tag{4.12}$$

That the Ricci curvature be nonpositive implies $DX = 0$ so X is parallel. Hence if X is not identically zero, it is nowhere vanishing. If the Ricci curvature is moreover somewhere negative, then (4.12) forces that X is identically zero. The rest of the argument is as before. □

Formally identical arguments, with the word negative changed to positive, yield proofs of Theorem 4.5.

Proof of Theorem 4.5. By Hodge theory, every first cohomology class is represented by a unique harmonic one-form X_i. A one-form is harmonic if and only if it is closed and coclosed. In (4.10) this yields

$$\tfrac{1}{2}\Delta|X|^2 = X^pX^qR_{pq} + |DX|^2 \geq X^pX^qR_{pq} \geq 0. \tag{4.13}$$

Hence $|X|^2$ is subharmonic. Since M is compact, this means $|X|^2$ is constant. In (4.13), this implies $0 = \frac{1}{2}\Delta|X|^2 \geq |DX|^2 \geq 0$, so $|DX|^2 = 0$ and X is parallel. In particular, if X is nontrivial it is nowhere vanishing. If the Ricci curvature is somewhere positive and X is not identically zero, then (4.11) implies $0 = \frac{1}{2}\Delta|X|^2 \geq X^pX^qR_{pq} > 0$, a contradiction. In general, the a vector subbundle spanned by parallel vector fields can have dimension no more then n. □

Alternative proof of Theorem 4.5. If X_i harmonic, integrating (4.13) over M yields

$$-\int_M X^pX^qR_{pq}\, d\mathsf{vol}_g = \int_M |DX|^2\, d\mathsf{vol}_g. \tag{4.14}$$

That the Ricci curvature be nonnegative implies $DX = 0$ so X is parallel. Hence if X is not identically zero, it is nowhere vanishing. If the Ricci curvature is moreover somewhere positive, then (4.14) forces that X is identically zero. The rest of the argument is as before. □

Theorem 4.4 can be improved as follows.

Theorem 4.6. *Let (M, g) be an n-dimensional compact Riemannian manifold.*

1. *If the Ricci curvature is nonpositive and negative at some point of M, then there is no nontrivial conformal Killing field. Consequently there are no one-parameter groups of isometries of g and the isometry group of g is a finite group.*
2. *If the Ricci curvature is nonpositive, every conformal Killing field is parallel and Killing, and the dimension of the space of Killing fields is no greater than n.*

Proof. Let X be a conformal Killing field. Then $n\mathfrak{L}_Xg = \operatorname{div}(X)g$, so (4.9) yields

$$\tfrac{1}{2}\Delta|X|^2 = \tfrac{1-n}{n}X^pD_p\operatorname{div}(X) - X^pX^qR_{pq} + |DX|^2, \tag{4.15}$$

and integrating (4.15) by parts yields

$$\begin{aligned}\int_M X^pX_qR_{pq}d\,\mathsf{vol}_g &= \int_M |DX|^2 d\,\mathsf{vol}_g + \tfrac{1-n}{n}\int_M X^pD_p\operatorname{div}(X)d\,\mathsf{vol}_g\\ &= \int_M |DX|^2 d\,\mathsf{vol}_g + \tfrac{n-1}{n}\int_M \operatorname{div}(X)^2 d\,\mathsf{vol}_g.\end{aligned} \tag{4.16}$$

That the Ricci curvature be nonpositive implies $DX = 0$ and $\operatorname{div}(X) = 0$ so X is parallel and Killing. The remaining claims follow as in the proof of Theorem 4.4. □

4.4. **Discussion of Bochner formulas.** Let $E \to M$ be a vector bundle on the n-dimensional Riemannian manifold (M, g), and suppose E is equipped with a fiberwise inner product that will be indicated $\langle s, t\rangle_E$ for $s, t \in \Gamma(E)$. Define a bilinear pairing $(\,\cdot\,,\,\cdot\,)_E : \Gamma(E) \times \Gamma(E) \to \mathbb{R}$ by

$$(s, t)_E = \int_M \langle s, t\rangle_E\, \mathsf{vol}_g. \tag{4.17}$$

In general at least one of s and t needs to have compact support for this to make sense. In what follows it will be supposed that such expressions make sense (that is, M is compact, or enough of the considered sections are compactly supported that the expressions considered make sense).

If P is a differential operator sending sections of E to sections of some other vector bundle $F \to M$, also equipped with a fiberwise inner product, the **formal adjoint** P^* of P is the linear operator $P^* : \Gamma(F) \to \Gamma(E)$ defined by

$$(Pe, f)_F = (e, P^* f)_E \tag{4.18}$$

for $e \in \Gamma(E)$ and $f \in \Gamma(F)$. Informally, P^* is defined by integration by parts. In general the formal adjoint of a differential operator P will again be a differential operator. In that case its expression as a differential operator, once computed, can be taken as its definition, and this permits extending its definition to nonintegrable sections.

Note that what is P^* depends on the choice of inner products on E and F and on the choice of volume form. What the so-to-speak right choices are is context dependent.

Example 4.7. Let E be a bundle of tensors. Then $F = T^*M \otimes E$ is also a bundle of tensors and the Levi-Civita connection D of the Riemannian metric g on M is an operator $D : \Gamma(E) \to Ga(F)$. Endowing E and F with the inner products determined by complete contraction, there results

$$(\nabla\alpha, \beta) = -(\alpha, \operatorname{div}\beta), \tag{4.19}$$

where $\operatorname{div}(\beta)$ equals the contraction of $D\beta$ on the index of D and the first index of β. Hence $D^* = -\operatorname{div}$ and $\Delta = -D^*D = -D^pD_p$.

Note that PP^* maps $\Gamma(E) \to \Gamma(E)$ and P^*P maps $\Gamma(F) \to \Gamma(F)$. The most interesing cases are when there is given a sequence of vector bundles and differential operators between them. For example, the sequence of exterior powers $\Omega^k(T^*M)$ and the exterior derivative $P_k = d : \Gamma(\Omega^k(T^*M)) \to \Gamma(\Omega^{k+1}(T^*M))$. The subscript indicates the dependence on k. The inner product on $\Gamma(\Omega^k(T^*M))$ is defined by

$$(\alpha, \beta)_k = \tfrac{1}{k!}\int_M \alpha^{i_1\dots i_k}\beta_{i_1\dots i_k}\,\mathsf{vol}g. \tag{4.20}$$

The factor of $(k!)^{-1}$ reflects that the inner product $\langle \cdot\,,\,\cdot \rangle_{\Omega^k(T^*M)}$ used here is that induced by g rather than that determined by complete contraction. In this case the operator P_k^* is given by $\alpha_{i_1\dots i_k} \to -D^p\alpha_{pi_1,\dots,i_{k-1}}$. The reason for defining $(\cdot\,,\,\cdot)_k$ as in (4.20) is motivated by (4.23) below. An expression such as $\square = P_{k+1}^*P_k + P_{k-1}P_k^*$ makes sense. This particular operator $\square$ is the **Hodge Laplacian**. Were a different inner product used in (4.20), the corresponding adjoint operators would be $\tilde{P}_k^* = c_kP_k^*$ instead of P_k^* for some coefficients c_k depending on k, and $\square$ would have the less aesthetic form $c_{k+1}\tilde{P}_{k+1}^*P_k + c_kP_{k-1}\tilde{P}_k^*$.

For another example of consider the sequence of symmetric powers $S^k(T^*M)$ and the operator $\mathcal{L}_k(\omega)_{i_1\dots i_{k+1}} = (k+1)D_{(i_1}\omega_{i_2\dots i_{k+1})}$ for $\omega \in \Gamma(S^k(T^*M))$. When $k = 1$, $\mathcal{L}(X)$ is the Killing operator considered before. The inner product on $\Gamma(S^k(T^*M))$ is defined by

$$(\alpha, \beta) = \tfrac{1}{k!}\int_M \alpha^{i_1\dots i_k}\beta_{i_1\dots i_k}\,\mathsf{vol}g. \tag{4.21}$$

The formal adjoint $\mathcal{L}_k^*$ is given by $\mathcal{L}_k^*(\omega)_{i_1\dots i_{k-1}} = -D^p\omega_{pi_1,\dots,i_{k-1}}$. When $k = 0$, there had to be understood that $\mathcal{L} = d$. In this case $\square = (-1)^k\mathcal{L}_{k-1}\mathcal{L}_k^* + \mathcal{L}_{k+1}^*\mathcal{L}_k$.

Writing $\Delta = D^pD_p$, the formulas (4.7) and (4.8) have the forms

$$\begin{aligned} D^*DX = -\Delta X &= -\mathcal{L}\mathcal{L}^*(X) + \mathcal{L}^*\mathcal{L}(X) + \mathcal{R}(X) \\ &= \square X + \mathcal{R}(X), \end{aligned} \tag{4.22}$$

where $\mathcal{R}(X)_i = R_{ip}X^p$. For $X \in \Gamma(T^*M)$,

$$D^*DX = -\Delta X = dd^*X + d^*dX - \mathcal{R}(X) = \square X - \mathcal{R}(X), \tag{4.23}$$

where $\mathcal{R}(X)_i = R_{ip}X^p$.

Formulas like (4.22) and (4.23) are called Bochner-Weitzenböck formulas. The general questions of for what vector bundles there exist such formulas and how to choose the coefficients when such formulas exist are essentially representation theoretic, are discussed in [22] and [38] and the references in these papers.

4.5. **Refined Kato inequality for Killing fields.** In section 4.6 it is shown how to extend Theorem 4.4 to complete Riemannian manifolds with nonpositive Ricci curvature.

Although this extension is interesting in its own right, the true objective is to introduce a new technique that will be needed again later, namely the use of refined Kato inequalities. Here this is described in one of the simplest special, cases, namely for Killing fields.

The classical Kato inequality is given by the following theorem.

Theorem 4.8. *Let (M, g) be a Riemannian manifold and $E \to M$ a smooth vector bundle associated with the frame bundle of M by a representation of the orthogonal group, equipped with the metric induced by g. Let ∇ be a connection on E preserving the metric. If s is any section of E then*

$$|\nabla s| \geq |d|s||, \tag{4.24}$$

on the set where s does not vanish. Moreover, there is equality in (4.24) if and only if there is a one-form γ such that $\nabla s = \gamma \otimes s$ wherever s does not vanish.

Proof. This proof follows section 2 of [9]. Work on the open set $U \subset M$ where s does not vanish. For any $X \in \Gamma(TM)$, $d|s|^2(X) = 2\langle \nabla_X s, s\rangle$, so $|d|s|^2| = 2|\langle \nabla s, s\rangle|$, where the norms are norms of 1-forms on M. Endow $T^*M \otimes E$ with the tensor product metric. By the Cauchy-Schwarz inequality, $|\langle \nabla_X s, s\rangle| \leq |\nabla_X s||s|$ with equality for a particular X if and only if $\nabla_X s = \gamma(X)s$ for some function $\gamma(X)$. Hence equality holds for all X if and only if there is a one-form γ on U such that $\nabla s = \gamma \otimes s$. Since $|\langle \nabla_X s, s\rangle| \leq |\nabla_X s||s|$ holds for all X, there holds $|\langle \nabla s, s\rangle| \leq |\nabla s||s|$. There follows

$$2|s||d|s|| = |d|s|^2| = 2|\langle \nabla s, s\rangle| \leq 2|\nabla s||s|, \tag{4.25}$$

with equality if and only if there is a one-form γ on U such that $\nabla s = \gamma \otimes s$. □

Remark 4.9. If M is assumed to be a spin manifold, then Theorem 4.8 makes sense for a vector bundle E associated with the frame bundle of M via a representation of the spin group. The proof is the same. This extension is useful in applications.

A **refined Kato inequality** is an inequality of the form $\tau|\nabla s| \geq |d|s||$ for some $\tau > 1$. Such an inequality cannot be true for arbitrary sections of E, but it can hold for sections of E in the kernel of some elliptic operator. This idea will be developed in more generality later, but one of the simplest examples is given by Lemma 4.10.

Lemma 4.10 (Refined Kato inequality for Killing fields)**.** *If X is a Killing field on the Riemannian manifold (M, g), then*

$$|DX|^2 \geq 2|d|X||^2, \tag{4.26}$$

wherever X does not vanish.

Proof. By assumption $D_i X_j = -D_j X_i$ and $\operatorname{div}(X) = 0$. Work on the open set U of M where X does not vanish. Let Z be a vector field on U. Then

$$\tfrac{1}{2} Z^p d_p |X|^2 = Z^p X^q D_p X_q = Z^p X^q D_{[p} X_{q]} = Z^{[p} X^{q]} D_{[p} X_{q]} = Z^{[p} X^{q]} D_p X_q. \tag{4.27}$$

Let $\sigma_{ij} = Z_{[i} X_{j]}$. Then $|\sigma|^2 = \frac{1}{2}(|X|^2|Z|^2 - (Z^p X_p)^2)$. Hence, by (4.27) and the Cauchy-Schwarz inequality,

$$\tfrac{1}{2} Z^p d|X|^2 = Z^{[p} X^{q]} D_p X_q \leq |\sigma||DX| = \tfrac{1}{\sqrt{2}}\sqrt{(|X|^2|Z|^2 - (Z^p X_p)^2)}|DX| \leq \tfrac{1}{\sqrt{2}}|X||Z||DX|. \tag{4.28}$$

Hence $|Z^p d_p|X|^2| \leq \sqrt{2}|X||Z||DX|$ for all Z. This implies $2|X||d|X|| = |d|dX|^2| \leq \sqrt{2}|X||DX|$, which yields (4.26). □

Remark 4.11. When E is the associated vector bundle of a representation of the orthogonal or spin group, the constant in the refined Kato inequality is determined representation theoretically. This is explained in detail in the papers [10] and [7]. The survey article [10] is particularly recommended as an introduction.

4.6. **Killing fields on complete Riemannian manifolds.** In this section it is shown how to extend Theorem 4.4 to complete Riemannian manifolds with nonpositive Ricci curvature.

Lemma 4.12. *On a complete n-dimensional Riemannian manifold (M, g) any Killing field X satisfies*

$$\Delta \log |X| \geq -|\mathsf{Ric}| \tag{4.29}$$

on the open subset of M where X does not vanish. In particular:

(1) *If the Ricci curvature of g satisfies $R_{ij} \leq -\kappa(n-1)$ for some function $\kappa \geq 0$, then*

$$\Delta \log |X| \geq \kappa(n-1) \tag{4.30}$$

on the open subset of M where X does not vanish, and $\log |X|$ is a subharmonic function on M.

(2) *If a complete Riemannian manifold (M, g) admits a nontrivial Killing field X of constant norm, then its Ricci curvature is nonnegative along X, that is $R_{ij}X^iX^j \leq 0$. In particular, a complete Riemannian manifold of nonpositive Ricci curvature that has Ricci curvature negative at some point admits no nontrivial Killing field of constant norm.*

Proof. By (4.9) and Lemma 4.10

$$\tfrac{1}{2}\Delta|X|^2 = |DX|^2 - X^pX^qR_{pq} \geq 2|d|X||^2 + \kappa(n-1)|X|^2. \tag{4.31}$$

For any nonnegative function $v \in C^\infty(M)$ there holds

$$\Delta \log v = v^{-1}\Delta v - v^{-2}|dv|^2, \tag{4.32}$$

on the open subset where $v > 0$.

On the open subset U of M where X does not vanish, the function $\log |X|$ is smooth and, by (4.32) and (4.31), there holds

$$\begin{aligned} \Delta \log |X| &= \tfrac{1}{2}\Delta \log |X|^2 = \tfrac{1}{2}|X|^{-2}\Delta|X|^2 - \tfrac{1}{2}|X|^{-4}|d|X|^2|^2 \\ &= \tfrac{1}{2}|X|^{-2}\left(\Delta|X|^2 - 4|d|X||^2\right) \geq -|X|^{-2}X^pX^qR_{pq} \geq -|\mathsf{Ric}|, \end{aligned} \tag{4.33}$$

the final step by the Cauchy-Schwarz inequality. In particular, if $R_{ij} \leq -\kappa(n-1)$ for some nonnegative function κ, then $\Delta \log |X| \geq \kappa(n-1) \geq 0$ and $\log |X|$ is a subharmonic function on M. Finally, if X is a nontrivial Killing field of constant norm, then (4.33) implies $0 \leq X^pX^qR_{pq}$. □

Remark 4.13. Note that the estimation (4.33) fails if one uses only the Kato inequality and not the refined version.

Remark 4.14. In applying a result like Lemma 4.33, it is important to keep in mind the following simple example: the function $\log |x|$ on $\mathbb{R}^n$ is subharmonic.

Theorem 4.15. *On a complete n-dimensional Riemannian manifold (M, g) with Levi-Civita connection D and having Ricci curvature satisyfing $R_{ij} \leq -\kappa(n-1)$ for some function $\kappa \geq 0$, if the norm $|X|$ of a Killing field X attains a local maximum at some point of M, then $|X|$ is constant and X is parallel. Moreover, if $\kappa \geq 0$ is somewhere positive, then X is identically zero.*

Proof. If X is not identically zero, the value of $|X|$ at a weak local maximum must be positive, so the point at which it is attained must lie in the open subset U of M on which X does not vanish, where $\log|X|$ is subharmonic. By the strong maximum principle (Theorem 5.8, discussed in section 5), X must be constant on the connected component of U containing X. By a theorem of S. Kobayashi in [25], every connected component of the zero set of a Killing field on a Riemannian manifold is a totally geodesic submanifold (possibly a point) of even codimension. By the theorem of Kobayashi, U is in fact connected. By continuity $|X|$ must be constant on all of M. In this case, X is parallel by (4.31). If the constant is nonzero then by (4.30) of Lemma 4.12, $0 \geq \kappa(n-1)$ so κ must be identically zero. □

The following example shows that there is a complete Riemannian manifold of nonnegative Ricci curvature that supports a Killing field that is not parallel, namely flat Euclidean space.

Example 4.16. A vector field on the n-dimensional real vector space $\mathbb{V}$ preserves the Levi-Civita connection D of the Euclidean metric δ_{ij} if and only if it has the form

$$X = x^i A_i{}^p \partial_{x^p} + b^p \partial_{x^p}, \tag{4.34}$$

where $x^1, \dots, x^n$ are coordinate functions such that the coordinate vector fields $\partial_{x^1}, \dots, \partial_{x^n}$ are parallel, and $A_i{}^j$ and b^j are constant. Since $D_i X^j = A_i{}^j$, there holds $D_{(i}X_{j)} = 0$ if and only if $A_i{}^j$ is skew-symmetric with respect to δ_{ij}, that is $A_{ij} = -A_{ji}$. Equivalently, X is an infinitesimal Euclidean motion. In this case

$$|X|^2 = -x^p x^q A_p{}^a A_{aq} + 2x^p A_p{}^a b_p + b^p b_p. \tag{4.35}$$

Consequently,

$$d_i|X|^2 = -2x^q A_i{}^a A_{aq} + 2A_i{}^p b_p, \qquad D_i d_j |X|^2 = -2A_i{}^p A_{pj}. \tag{4.36}$$

Since $A_i{}^p A_{pj} = (A \circ A)_{ij}$ where $A \circ A$ is the composition of endomorphisms, and the square of an antisymmetric endomorphism is nonpositive, the Hessian $D_i d_j |X|^2$ is nonnegative. On the other hand, as at a local maximum of $|X|^2$ the Hessian must be nonpositive, at such a local maximum the Hessian must be identically zero. This means $A_i{}^j$ is identically zero, and so X is purely translational, with constant norm $|b|^2$.

4.7. Refined Kato inequality for harmonic one-forms.

Lemma 4.17. *On an n-dimensional Riemannian manifold (M, g), let X be a closed one-form (so $D_{[i}X_{j]} = 0$). Define* $\operatorname{div}(X) = D^p X_p$. *Wherever X does not vanish there hold*

$$\begin{aligned} |DX|^2 &\geq \tfrac{1}{n-1}(\operatorname{div}(X))^2 - \tfrac{1}{n-1}\operatorname{div}(X)|X|^{-2}X^p d_p|X|^2 + \tfrac{n}{4(n-1)}|X|^{-2}|d|X|^2|^2 \\ &= \tfrac{1}{n-1}(\operatorname{div}(X))^2 - \tfrac{1}{n-1}(\operatorname{div}(X))|X|^{-2}X^p d_p|X|^2 + \tfrac{n}{n-1}|d|X||^2, \end{aligned} \tag{4.37}$$

and

(4.38)
$$\tfrac{1}{2}\Delta|X|^2 \geq \tfrac{1}{n-1}(\operatorname{div}(X))^2 - \tfrac{1}{n-1}\operatorname{div}(X)|X|^{-2}X^p d_p|X|^2 + X^p \operatorname{div}(X)_p + \tfrac{n}{n-1}|d|X||^2 + X^p X^q R_{pq}.$$

Proof. Let Z^i be any vector field. Consider the trace-free symmetric tensor $\sigma_{ij} = Z_{(i}X_{j)} - \frac{1}{n}Z^p X_p g_{ij}$. There holds

$$|\sigma|^2 = Z^i X^j \sigma_{ij} = Z^i X^j (Z_{(i}X_{j)} - \tfrac{1}{n}Z^p X_p g_{ij}) = \tfrac{1}{2}|Z|^2|X|^2 + \tfrac{n-2}{2n}(Z^p X_p)^2 \leq \tfrac{n-1}{n}|Z|^2|X|^2, \tag{4.39}$$

the inequality by the Cauchy-Schwarz inequality. Consequently,

$$\begin{aligned}\tfrac{1}{2}Z^p d_p|X|^2 &= Z^i X^j D_i X_j = Z^i X^j D_{(i}X_{j)}\\ &= Z^i X^j (D_{(i}X_{j)} - \tfrac{1}{n}\operatorname{div}(X) g_{ij} + \tfrac{1}{n}\operatorname{div}(X) g_{ij})\\ &= \sigma^{ij}(D_{(i}X_{j)} - \tfrac{1}{n}\operatorname{div}(X) g_{ij}) + \tfrac{1}{n}\operatorname{div}(X)(Z^p X_p)\\ &= \sigma^{ij}(D_i X_j - \tfrac{1}{n}\operatorname{div}(X) g_{ij}) + \tfrac{1}{n}\operatorname{div}(X)(Z^p X_p)\\ &\leq |\sigma| \left(|DX|^2 - \tfrac{1}{n}\operatorname{div}(X)^2\right)^{1/2} + \tfrac{1}{n}\operatorname{div}(X)(Z^p X_p)\\ &\leq \sqrt{\tfrac{n-1}{n}}|Z||X| \left(|DX|^2 - \tfrac{1}{n}\operatorname{div}(X)^2\right)^{1/2} + \tfrac{1}{n}\operatorname{div}(X)(Z^p X_p).\end{aligned} \tag{4.40}$$

In what follows it is important not to discard any terms. Equivalently,

$$Z^p\left(\tfrac{1}{2}d_p|X|^2 - \tfrac{1}{n}\operatorname{div}(X)X_p\right) \leq \sqrt{\tfrac{n-1}{n}}|Z||X|\left(|DX|^2 - \tfrac{1}{n}\operatorname{div}(X)^2\right)^{1/2}. \tag{4.41}$$

Since (4.41) is true for all vector fields Z, it implies

$$\left|\tfrac{1}{2}d|X|^2 - \tfrac{1}{n}\operatorname{div}(X)X\right| \leq \sqrt{\tfrac{n-1}{n}}|X|\left(|DX|^2 - \tfrac{1}{n}\operatorname{div}(X)^2\right)^{1/2}. \tag{4.42}$$

Squaring (4.42) and rearranging terms yields (4.37).

From

$$\Delta X_i = D^p D_p X_i = D^p D_i X_p = D_i D^p X_p - R^p{}_{ip}{}^q X_q = \operatorname{div}(X)_i + X^p R_{ip}, \tag{4.43}$$

there follows

$$\begin{aligned}\tfrac{1}{2}\Delta|X|^2 &= \tfrac{1}{2}D^p D_p(X^i X_i) = D^p(X^i D_p X_i) = |DX|^2 + X^i \Delta X_i\\ &= |DX|^2 + X^i \operatorname{div}(X)_i + X^p X^q R_{pq}.\end{aligned} \tag{4.44}$$

Substituting (4.37) in (4.44) yields (4.38). □

Taking $X = df$ in Lemma 4.17 yields immediately:

Lemma 4.18. *On an n-dimensional Riemannian manifold (M, g), for any $f \in C^\infty(M)$, wherever df does not vanish there hold*

$$\begin{aligned}|Ddf|^2 &\geq \tfrac{1}{n-1}(\Delta f)^2 - \tfrac{1}{n-1}(\Delta f)|df|^{-2} f^p d_p|df|^2 + \tfrac{n}{4(n-1)}|df|^{-2}|d|df|^2|^2\\ &= \tfrac{1}{n-1}(\Delta f)^2 - \tfrac{1}{n-1}(\Delta f)|df|^{-2} f^p d_p|df|^2 + \tfrac{n}{n-1}|d|df||^2,\end{aligned} \tag{4.45}$$

and

$$\tfrac{1}{2}\Delta|df|^2 \geq \tfrac{1}{n-1}(\Delta f)^2 - \tfrac{1}{n-1}(\Delta f)|df|^{-2} f^p d_p|df|^2 + f^p(\Delta f)_p + \tfrac{n}{n-1}|d|df||^2 + f^p f^q R_{pq}, \tag{4.46}$$

where, for $v \in C^\infty(M)$, v_i means dv_i.

Let X be a one-form. From the Cauchy-Schwarz inequality

$$\tfrac{1}{2}Z^p d_p|X|^2 = Z^i X^j D_i X_j \leq |Z||X||DX| \tag{4.47}$$

for all vector fields Z, which implies $|d|X||^2 \leq |DX|^2$. This is the usual Kato inequality for one-forms. If $dX = 0$ and $\operatorname{div}(X) = 0$, then (4.45) yields the following stronger result:

Theorem 4.19 (Refined Kato inequality for harmonic one-forms)**.** *On an n-dimensional Riemannian manifold (M, g), let X be a harmonic one-form, that is a closed one-form (so $D_{[i}X_{j]} = 0$) satisfying* $\operatorname{div}(X) = D^p X_p = 0$. *Where X does not vanish there holds*

$$\tfrac{n}{n-1}|d|X||^2 \leq |DX|^2. \tag{4.48}$$

Proof. This follows from Lemma 4.17 when $\operatorname{div}(X) = 0$. □

This is the refined Kato inequality for harmonic one-forms. In particular (4.48) holds when $X = df$ and $\Delta f = 0$. This refinement is essential for obtaining the sharp gradient estimate for harmonic functions.

Lemma 4.20. *Let $n > 2$. On a complete n-dimensional Riemannian manifold (M, g) any harmonic one-form X satisfies*

$$\Delta |X|^{\frac{n-2}{n-1}} \geq \tfrac{n-2}{n-1}(|X|^{-2}X^pX^qR_{pq})|X|^{\frac{n-2}{n-1}} \tag{4.49}$$

on the open subset of M where X does not vanish. In particular, if the Ricci curvature of g satisfies $R_{ij} \leq \kappa(n-1)$ for some function $\kappa \geq 0$, then

$$\Delta |X|^{\frac{n-2}{n-1}} \geq \kappa(n-2)|X|^{\frac{n-2}{n-1}} \tag{4.50}$$

on the open subset of M where X does not vanish. In particular, if $\kappa \geq 0$, then $|X|^{\frac{n-2}{n-1}}$ is a subharmonic function on M.

4.8. **Identities for eigenfunctions of the Laplacian.** Suppose $\Delta u = -\lambda u$. In (4.42) there results

$$\tfrac{4(n-1)}{n}|du|^2\left(|Ddu|^2 - \tfrac{\lambda^2}{n}u^2\right) \geq 4\left|\tfrac{1}{2}d|du|^2 + \tfrac{\lambda}{n}udu\right|^2 = \left|d(|du|^2 + \tfrac{\lambda}{n}u^2)\right|^2. \tag{4.51}$$

This motivates consideration of the associated function $|du|^2 + \frac{\lambda}{n}u^2$. It also shows $|Ddu|^2 \geq \frac{\lambda^2}{n}u^2$ whenever $du \neq 0$. In fact this inequality is true everywhere, for any function f, for taking $u = f$ in (9.5) yields $|Ddu|^2 \geq \frac{\lambda^2}{n}u^2$ when $\Delta u = -\lambda u$.

Corollary 4.21. *Let (M, g) be an n-dimensional Riemannian manifold and $u \in C^\infty(M)$ a function satisfying $\Delta u = \lambda u$ for some $\lambda \in \mathbb{R}$ and bounded from below. For any $a \in \mathbb{R}$ such that $u + a$ is positive on M let $v = \log(u + a)$. Then*

$$\tfrac{1}{2}\Delta|dv|^2 \geq \tfrac{1}{n-1}|dv|^4 - \tfrac{2(n-2)}{n-1}|dv|v^pd_p|dv| + \tfrac{n}{n-1}|d|dv||^2 + v^pv^qR_{pq} \tag{4.52}$$

$$+ \lambda\left(\tfrac{n+1}{n-1}ae^{-v} - \tfrac{2}{n-1}\right)|dv|^2 + \tfrac{\lambda^2}{n-1}(ae^{-v} - 1)^2 + \tfrac{2\lambda}{n-1}(ae^{-v} - 1)|dv|^{-1}v^pd_p|dv|. \tag{4.53}$$

In particular, if u is harmonic,

$$\begin{aligned}\tfrac{1}{2}\Delta|dv|^2 &\geq \tfrac{1}{n-1}|dv|^4 - \tfrac{2(n-2)}{n-1}|dv|v^pd_p|dv| + \tfrac{n}{n-1}|d|dv||^2 + v^pv^qR_{pq}\\ &\geq \tfrac{4-n}{n}|dv|^4 + \tfrac{n}{n-1}\left(|d|dv|| - \tfrac{n-2}{n}|dv|^2\right)^2 + v^pv^qR_{pq}.\end{aligned} \tag{4.54}$$

Remark 4.22. Equation (4.54) is equation 2.4 of [28].

Proof. Let $u \in C^\infty(M)$ satisfy $\Delta u = \lambda u$. Suppose that $a \in \mathbb{R}$ is such that $u + a$ is positive on M. (If M is compact, such an a can always be chosen; simply choose a greater than the absolute value of the minimum of u. If M is noncompact, the existence of such an a imposes an additional assumption on u, namely that it is bounded from below). Define $v = \log(u + a)$. Then

$$\Delta v = \lambda - a\lambda e^{-v} - |dv|^2. \tag{4.55}$$

Taking $f = v$ in (4.46) yields

$$\begin{aligned}\tfrac{1}{2}\Delta|dv|^2 &\geq \tfrac{1}{n-1}(|dv|^2 + \lambda(ae^{-v} - 1))^2 + \tfrac{1}{n-1}(|dv|^2 + \lambda(ae^{-v} - 1))|dv|^{-2}v^pd_p|dv|^2\\ &\quad + v^pd_p|dv|^2 - \lambda ae^{-v}|dv|^2 + \tfrac{n}{n-1}|d|dv||^2 + v^pv^qR_{pq},\\ &= \tfrac{1}{n-1}|dv|^4 - \tfrac{n-2}{n-1}v^pd_p|dv|^2 + \tfrac{n}{n-1}|d|dv||^2 + v^pv^qR_{pq}\\ &\quad + \lambda\left(\tfrac{2}{n-1}(ae^{-v} - 1) - ae^{-v}\right)|dv|^2 + \tfrac{\lambda^2}{n-1}(ae^{-v} - 1)^2\\ &\quad + \tfrac{\lambda}{n-1}(ae^{-v} - 1)|dv|^{-2}v^pd_p|dv|^2,\end{aligned} \tag{4.56}$$

and simplifying the final terms gives (4.52). The first inequality of (4.54) is the special case of (4.52) when $\lambda = 0$, while the second inequality of (4.52) follows from

$$\begin{aligned} &|dv|^4 - (n-2)v^p d_p|dv|^2 + n|d|dv||^2 \\ &= |dv|^4 - 2(n-2)|dv|v^p d_p|dv| + n|d|dv||^2 \\ &\geq |dv|^4 - 2(n-2)|dv|^2|d|dv| + n|d|dv||^2 \\ &= \tfrac{(4-n)(n-1)}{n}|dv|^4 + n\left(|d|dv|| - \tfrac{n-2}{n}|dv|^2\right)^2. \end{aligned} \tag{4.57}$$

□

Lemma 4.23 (Lemma 23.4 in [27]). *Let (M, g) be an n-dimensional Riemannian manifold with Ricci curvature R_{ij} satisfying $R_{ij} \geq -\kappa(n-1)g_{ij}$ for some $\kappa \in C^\infty(M)$. If $f \in C^\infty(M)$ is harmonic, then, wherever df is not zero there holds*

$$\Delta|df|^\lambda \geq -(n-1)\lambda\kappa|df|^\lambda \tag{4.58}$$

for any $\lambda \geq \frac{n-2}{n-1}$. In particular,

$$\Delta|df|^{\frac{n-2}{n-1}} \geq -(n-2)\kappa|df|^{\frac{n-2}{n-1}}. \tag{4.59}$$

Proof. For any nonnegative function $v \in C^\infty(M)$ and any $\beta \neq 0$ there holds

$$\Delta v^\beta = \beta v^{\beta-1}\left(\Delta v + (\beta-1)v^{-1}|dv|^2\right), \tag{4.60}$$

on the open subset where $v > 0$. Taking $v = |df|^2$ in (4.60) and using (4.45) yields

$$\begin{aligned} \tfrac{1}{2\beta}\Delta|df|^{2\beta} &= |df|^{2(\beta-1)}\left(\tfrac{1}{2}\Delta|df|^2 + 2(\beta-1)|d|df||^2\right) \\ &\geq |df|^{2(\beta-1)}\left(\left(\tfrac{n}{n-1} + 2(\beta-1)\right)|d|df||^2 + f^p f^q R_{pq}\right) \\ &\geq |df|^{2(\beta-1)}\left(\left(2\beta - \tfrac{n-2}{n-1}\right)|d|df||^2 - (n-1)\kappa|df|^2\right) \\ &= \left(2\beta - \tfrac{n-2}{n-1}\right)|df|^{2(\beta-1)}|d|df||^2 - (n-1)\kappa|df|^{2\beta}, \end{aligned} \tag{4.61}$$

valid wherever df does not vanish. If $\lambda = 2\beta \geq \frac{n-2}{n-1}$, (4.61) yields (4.58). In particular, taking $2\beta = \frac{n-2}{n-1}$ in (4.61) yields (4.59). □

A more refined result can be obtained by working with the logarithm of f instead of f.

Theorem 4.24. *Let (M, g) be an n-dimensional Riemannian manifold with Ricci curvature satisfying $R_{ij} \geq -(n-1)\kappa$ for some $\kappa \in C^\infty(M)$. Let $f \in C^\infty(M)$ be a harmonic function that is bounded from below. For any $a \in \mathbb{R}$ such that $f + a$ is positive on M let $v = \log(f + a)$. Then, for any $\beta > \frac{n-2}{2(n-1)}$, there holds*

$$\tfrac{1}{2}\Delta|dv|^{2\beta} \geq \tfrac{\beta(2\beta-(n-2))}{2(n-1)\beta-(n-2)}|dv|^{2\beta+2} - (n-1)\beta\kappa|dv|^{2\beta}. \tag{4.62}$$

Proof. By (4.54), if $\beta > \frac{n-2}{2(n-1)}$,

(4.63)
$$
\begin{aligned}
\tfrac{n-1}{2\beta}\Delta|dv|^{2\beta} &= (n-1)|dv|^{2(\beta-1)}\left(\tfrac{1}{2}\Delta|dv|^2 + 2(\beta-1)|d|dv||^2\right)\\
&\geq |dv|^{2(\beta-1)}\left((n-1)v^pv^qR_{pq} + |dv|^4 - 2(n-2)|dv|v^pd_p|dv| + (2(n-1)\beta-(n-2))|d|dv||^2\right)\\
&\geq |dv|^{2(\beta-1)}\left((n-1)v^pv^qR_{pq} + |dv|^4 - 2(n-2)|dv|^2|d|dv|| + (2(n-1)\beta-(n-2))|d|dv||^2\right)\\
&= |dv|^{2(\beta-1)}\left((n-1)v^pv^qR_{pq} + \tfrac{(2\beta-(n-2))(n-1)}{2(n-1)\beta-(n-2)}|dv|^4\right)\\
&\quad + (2(n-1)\beta-(n-2))\left(|d|dv|| - \tfrac{n-2}{2(n-1)\beta-(n-2)}|dv|^2\right)^2|dv|^{2(\beta-1)}\\
&\geq \tfrac{(2\beta-(n-2))(n-1)}{2(n-1)\beta-(n-2)}|dv|^{2\beta+2} - (n-1)^2\kappa|dv|^{2\beta}.
\end{aligned}
$$

This yields (4.62). □

5. Maximum principle

The proof of Theorem 4.15 used the strong maximum principle. Since this is important in its own right, a digression is made to prove it now.

The weak maximum principle says that a subsolution of a second order elliptic operator attains its maximum over a bounded, connected, open set on the boundary of the open set. The strong (or Hopf) maximum principle says that a subsolution of a a second order elliptic operator on a connected open set has no local maximum in the interior of the open set unless it is constant. The strong maximum principle obviously implies the weak maximum principle. However, for either principle to hold some conditions have to imposed on the elliptic operator. The two principles are distinguished because the weak maximum principle is valid for a wider class of operators than is the strong maximum principle.

The monograph [21] of D. Gilbarg and N. Trudinger gives standard general formulations of the maximum principle as commonly used. The standard comprehensive references for maximum principles are M. Protter and H. Weinberger's [35] and P. Pucci and J. Serrin's [36].

Suppose given an open domain $U \subset M$ and $f \in C^2(U) \cap C^0(\bar{U})$. If f attains a local maximum at $p \in U$ then, because $df(p) = 0$ and g is positive definite, by (3.10),

$$(\Delta f)(p) = g^{ij}\tfrac{\partial^2 f}{\partial x^i \partial x^j}(p) \leq 0. \tag{5.1}$$

If f is strictly subharmonic in U, meaning $\Delta f > 0$ in U, then (5.1) yields a contradiction. If f is only subharmonic in U, meaning $\Delta f \geq 0$ in U, the same argument fails, because (5.1) does not yield a contradiction. However, the same conclusion holds. This is what is usually known as the Hopf maximum principle.

Let ∂ be the standard flat affine connection on $\mathbb{R}^n$. A second order linear differential operator

$$Lf = a^{ij}\partial_i df_j + b^i df_j \tag{5.2}$$

with $a^{ji} = a^{ij}$ a symmetric contravariant tensor and b^i a vector field, is **elliptic** at $p \in U$ if a^{ij} is positive definite at p. The operator L is **elliptic** in an open set $U \subset \mathbb{R}^n$ if L is elliptic at every point of U.

By (3.10), on any local coordinate chart the Laplacian of a Riemannian metric has the form (5.2) and is elliptic. Consequently, for any vector field $X \in \Gamma(TM)$ on a Riemannian manifold (M, g), the differential operator of the form

$$\mathcal{L}f = \Delta f + df(X) \tag{5.3}$$

is elliptic. Moreover, at a local maximum of f there holds

$$(\mathcal{L}f)(p) = (\Delta f)(p) \leq 0. \tag{5.4}$$

The Hopf maximum principle states that a subsolution of an elliptic operator of the form (5.2) that attains a local maximum at an interior point of an open region must be constant. Precisely:

Theorem 5.1 (Hopf maximum principle)**.** *Suppose L is a differential operator of the form* (5.1) *in a connected, open domain $U \subset \mathbb{R}^n$ equipped with the Euclidean metric. Suppose there are positive constants c_1, c_2, and c_3 such that the coefficients a^{ij} and b^i satisfy: $|b(x)| \leq c_1$ in U; for any vector v_i, $a^{ij}(x)v_iv_j \geq c_2|v|^2$ in U; and $|a^{ij}(x)| \leq c_3$ in U. If $f \in C^2(U)$ satisfies $Lf \geq 0$ in U and f attains a local maximum at some point in U, then f is constant on U.*

Remark 5.2. For the proof of Theorem 5.1 as stated, see [21]. In [8], Calabi proves theorem 5.1 under more general hypotheses: he allows f to upper semicontinuous and to satisfy the inequality $Lf \geq 0$ in a weak sense.

Here the conclusion of theorem 5.1 is needed only for the special case where L is the Laplacian of a Riemannian metric. On the other hand, it is useful to have a version of the Hopf maximum principle valid for a certain kind of weak subharmonic function. Such a generalization was found by E. Calabi in [8]. The proof will be given following section 6 of J. Eschenburg and E. Heintze's [16] as modified in section 7 of J. Cheeger's [11]. The proof of [16] is repeated, with some novel typographical errors, in [4].

Definition 5.1. *Let f be a function continuous on an open set U. A* ***lower barrier*** *for f at $x_0 \in U$ is a function g, C^2 in some open neighborhood V of x_0 contained in U, such that $g(x_0) = f(x_0)$ and $f \geq g$ on V. An* ***upper barrier*** *for f at $x_0 \in U$ is a function g, C^2 in some open neighborhood V of x_0 contained in U, such that $g(x_0) = f(x_0)$ and $f \leq g$ on V.*

Example 5.3. For any $\delta \in (0, 1)$, the function $g(x) = |x|^2$ is a lower barrier for $f(x) = |x|$ at $0 \in B(0, \delta) \subset \mathbb{R}^n$.

Definition 5.2. *Let (M, g) be a connected Riemannian manifold and let $\phi \in C^0(M)$. A function $f \in C^0(U)$ satisfies $\Delta f \geq \phi$* ***in the barrier sense*** *if for every $p \in M$ and every $\epsilon > 0$ there is a lower barrier $f_{p,\epsilon}$ of f at p, C^2 in an open neighborhood $V_{p,\epsilon}$ of p, such that $\Delta f_{p,\epsilon} \geq \phi - \epsilon$ at p. In particular, a function $f \in C^0(M)$ is* ***subharmonic in the barrier sense*** *on M if for every $p \in M$ and every $\epsilon > 0$ there is a lower barrier $f_{p,\epsilon}$ of f at p, C^2 in an open neighborhood $V_{p,\epsilon}$ of p, such that $\Delta f_{p,\epsilon} \geq -\epsilon$ at p. More generally, for an operator $\mathcal{L}$ of the form* (5.3)*, a function $f \in C^0(M)$ is a* ***subsolution of $\mathcal{L}$ (or $\mathcal{L}$-subharmonic) in the barrier sense*** *on M if for every $p \in M$ and every $\epsilon > 0$ there is a lower barrier $f_{p,\epsilon}$ of f at p, C^2 in an open neighborhood $V_{p,\epsilon}$ of p, such that $\mathcal{L}f_{p,\epsilon} \geq -\epsilon$ at p.*

Remark 5.4. The definitions of $\Delta f \leq \phi$ in the barrier sense, superharmonic in the barrier sense, and $\mathcal{L}$-superharmonic in the barrier sense are formally identical, replacing upper by lower.

Remark 5.5. Since an open subset of a Riemannian manifold is again a Riemannian manifold, it makes sense to speak of a subsolution in the barrier sense on an open subset of a Riemannian manifold.

Lemma 5.6. *For an operator $\mathcal{L}$ of the form* (5.3) *on the Riemannian manifold (M, g), a function $f \in C^2(M)$ is $\mathcal{L}$-subharmonic in the barrier sense if and only if it is $\mathcal{L}$-subharmonic.*

Proof. If f is $\mathcal{L}$-subharmonic, then it is $\mathcal{L}$-subharmonic in the barrier sense trivially, for f can be used as a lower barrier for f on M. On the other hand, if f is $\mathcal{L}$-subharmonic in the barrier sense, then for every $p \in M$ and every $\epsilon > 0$ there is a lower barrier $f_{p,\epsilon}$ of f at p that is C^2 in an open neighborhood $V_{p,\epsilon}$ of p and such that $\mathcal{L}f_{p,\epsilon} \geq -\epsilon$ in $V_{p,\epsilon}$. Since $f - f_{p,\epsilon}$ has on $V_{p,\epsilon}$ its minimum at

p, there holds $0 \leq \Delta(f - f_{p,\epsilon})(p) = \mathcal{L}(f - f_{p,\epsilon})(p) \leq (\mathcal{L}f)(p) + \epsilon$. Hence $(\mathcal{L}f)(p) \geq -\epsilon$ for all $\epsilon > 0$ and so $(\mathcal{L}f)(p) \geq 0$. Thus f is $\mathcal{L}$-subharmonic in the usual sense. □

Example 5.7. The function $f(x) = |x|$ is subharmonic in the barrier sense on $\mathbb{R}^n$. Since $\Delta|x| = (n-1)|x|^{-1}$, f is subharmonic in the usual sense on the complement of the origin. For any $0 < \epsilon$ the function $g(x) = |x|^2$ is a lower barrier for f at 0 on the set $V_{0,\epsilon} = B(0,1)$, and $\Delta g = 2n \geq -\epsilon$, so f is subharmonic in the barrier sense.

Theorem 5.8. *If on a connected Riemannian manifold (M, g) a function $f \in C^0(M)$ is subharmonic in the barrier sense, then f attains no local maximum in M unless f is constant on M.*

This is a special case of the following more general theorem.

Theorem 5.9. *If on a connected Riemannian manifold (M, g) a function $f \in C^0(M)$ is $\mathcal{L}$-subharmonic in the barrier sense for a differential operator of the form $\mathcal{L} = \Delta f + df(X)$, $X \in \Gamma(TM)$, then f attains no local maximum in M unless f is constant on M.*

Proof. The proof follows that given for the Laplacian in section 6 of [16] as modified in the appendix to section 7 of [11]. The modifications necessary to accommodate the more general operator $\mathcal{L}$ are trivial.

If every point of M is contained in an open neighborhood on which f is constant, then f is constant on M. Hence if f is not constant on U there exists $p \in M$ such that f is not constant on any open neighborhood of p. Choose $\delta > 0$ so small that the open geodesic ball $B(p, \delta)$ is contained in a normal coordinate chart centered on p and there is some z contained in the boundary $\partial B(p, \delta)$ such that $f(z) < f(p)$. By continuity, $f(w) < f(p)$ for $w \in \partial B(p, \delta)$ sufficiently close to z, so $S = \{q \in \partial B(p, \delta) : f(q) = f(p)\}$ is a proper subset of the boundary $\partial B(p, \delta)$ bounded away from z. Let $x^1, \dots, x^n$ be normal coordinates centered on p and defined in some neighborhood of the closure of $B(p, \delta)$. Composing with a rotation around the origin, if necessary, it can be supposed that $z = (\delta, 0, \dots, 0)$. For a constant $k > 0$, define

$$\phi(x) = x^1 - k \sum_{j=2}^{n} (x^j)^2. \tag{5.5}$$

Since S is bounded away from z, there is a constant $C > 0$ such that $\sum_{j=2}^{n} (x^j)^2 \geq C$ on S. Hence $\phi \leq \delta - kC$ on S, so that if k is chosen larger than δC^{-1}, then ϕ is negative on S. Fix such a k. Note that $\phi(p) = 0$, and $d\phi$ does not vanish on the closure $\overline{B(p, \delta)}$ of $B(p, \delta)$.

For a constant $a > 0$, define $h = e^{a\phi} - 1 \in C^\infty(B)$. Then $h(p) = 0$, h is negative on S, and

$$\mathcal{L}h = e^{a\phi}(a\mathcal{L}\phi + a^2|d\phi|^2). \tag{5.6}$$

Since $d\phi$ is nowhere zero on $\overline{B(p, \delta)}$ and $\overline{B(p, \delta)}$ is compact, $|d\phi|^2$ is bounded from below on the closure of $B(p, \delta)$. Similarly, $\mathcal{L}\phi$ is bounded from below on $\overline{B(p, \delta)}$. Choosing a large enough, there results $\mathcal{L}h > 0$ on $\overline{B(p, \delta)}$. Fix such an a.

Since h is negative on S, for $\eta > 0$ sufficiently small, $f + \eta h$ satisfies $(f + \eta h)(w) < f(p)$ for all $w \in \partial B(p, \delta)$. Hence $f + \eta h$ attains its maximum over $B(p, \delta)$ at some interior point q of $B(p, \delta)$, and this maximum is at least $f(p)$. Since $f_{p,\epsilon} + \eta h$ is a lower barrier for $f + \eta h$ at q, it also has a local maximum at q. However, for sufficiently small ϵ, $\mathcal{L}(f_{p,\epsilon} + \eta h)(q) \geq -\epsilon + \eta \mathcal{L}h > 0$. By (5.4), this is inconsistent with $f_{p,\epsilon} + \eta h$ having a local maximum at q. □

6. Submanifolds

This section is a digression intended to recall the geometry of submanifolds. While this will be needed in various forms later, the most immediate application is that in section 7.5 there is discussed the mean curvature of geodesic spheres in a Riemannian manifold (level sets of the distance from a fixed point).

Let $i : \Sigma \to M$ be a smooth immersion. Let i^*TM be the pullback vector bundle over Σ, the fiber of which over $p \in \Sigma$ is $T_{i(p)}M$. The differential $Ti : T\Sigma \to TM$ induces an identification of $T\Sigma$ with a subbundle of $i^*(TM)$. A local section X of $T\Sigma$ is identified with the local section $p \to Ti(p)(X_p)$ of i^*TM. The normal bundle $Nl\Sigma$ of the immersion i is the fiberwise quotient of i^*TM by its subbundle identified with $T\Sigma$. Precisely, $Nl_p\Sigma = T_{i(p)}M/Ti(p)(T_p\Sigma)$. The notation ought to indicate the dependence of $Nl\Sigma$ on i, but it is standard that it does not. Often notation is further abused, and notation indicating the immersion i is completely omitted, a local section of $T\Sigma$ being identified with a local section of i^*TM.

The **conormal** bundle of the immersion $i : \Sigma \to M$ is the annihilator $\operatorname{Ann} T\Sigma$, viewed as a subbundle of i^*T^*M. An immersion is **coorientable** if its conormal bundle is orientable.

A torsion-free affine connection ∇ induces on i^*TM a connection, also denoted ∇, defined as follows. For an open neighborhood $U \subset \Sigma$ containing $p \in \Sigma$ and local sections $s \in \Gamma(U, i^*TM)$ and $X \in \Gamma(U, T\Sigma)$, let $\tilde{X}$ and $\tilde{Y}$ be vector fields defined on some open neighborhood in M containing $i(U)$ and such that $\tilde{X}_{i(p)} = Ti(p)(X_p)$ and $\tilde{Y}_{i(p)} = s_p$ for $p \in U$. Define $(\nabla_X s)_p = (\nabla_{\tilde{X}}\tilde{Y})_p$. It is straightforward to check that $\nabla_X s$ is well-defined (does not depend on the choices involved in its construction) as section over U of i^*TM, and that this defines a connection ∇ on i^*TM. This connection is called the connection **induced** by ∇ on i^*TM.

In a similar manner, ∇ induces a connection on the normal bundle. The details are left to the reader.

The **second fundamental form** $\Pi^{\nabla}(i)$ of the immersion i with respect to a torsion-free affine connection ∇ on M is the section $\Pi^{\nabla}(i)$ of $S^2(T^*\Sigma) \otimes Nl\Sigma$ defined as the projection onto $Nl\Sigma$ of $\nabla_X Y$ for $X, Y \in \Gamma(T\Sigma)$. Because $[X, Y]$ is tangent to Σ, its projection onto $Nl\Sigma$ is zero, so the symmetry of $\Pi^{\nabla}(i)$ follows from the fact that ∇ is torsion-free.

When i and ∇ are clear from context it is convenient to write simply Π in place of $\Pi^{\nabla}(i)$. In the abstract index notation, sections of tensor powers of distinct vector bundles are labeled using different alphabets. For examples sections of tensor powers of the normal bundle and its dual can be labeled using lowercase Greek indices, so that the second fundamental form is $\Pi_{ij}{}^{\alpha} = \Pi_{ji}{}^{\alpha}$. When Σ has condimension one its normal bundle is a line bundle and the Greek index is omitted, but it needs to be kept in mind that in this case Π_{ij} is not a tensor on M in the ordinary sense, rather it is a normal-bundle valued tensor.

An alternative way to view the second fundamental form is the following. The differential $Tf : \Sigma \to M$ can be viewed as a section of $T\Sigma \otimes f^*TM$. For $X, Y \in \Gamma(U, TM)$, define the **fundamental form** $\mathbb{D}Tf$ of the smooth map $f : \Sigma \to M$ by

$$(\mathbb{D}_X Tf)(Y) = \nabla_X Tf(Y) - Tf(\nabla_X Y), \tag{6.1}$$

where $\mathbb{D}$ is the connection induced on the tensor product $T\Sigma \otimes f^*TM$, and formally the right-hand side has to be interpreted in terms of local extensions of vector fields, as in the preceeding paragraphs. In abstract index notation, Tf can be written f_i^A, where uppercase Latin indices are used to label sections of f^*TM, and (6.1) defines $\mathbb{D}_i f_j{}^A$. Here $\mathbb{D}_i f_j{}^A$ is not quite the second fundamental form. Let $\sigma_A{}^{\alpha}$ be the projection from i^*TM to $Nl\Sigma$. Then

$$\Pi_{ij}{}^{A} = \sigma_A{}^{\alpha}\mathbb{D}_i f_j{}^A. \tag{6.2}$$

In the special case $M = \mathbb{R}$, the fundamental form $\mathbb{D}Tf$ is the Hessian of f with respect to ∇, usually written as ∇df. Hence the fundamental form $\mathbb{D}Tf$ is a generalization of the Hessian of a function $f : \Sigma \to \mathbb{R}$.

Let $i : \Sigma \to M$ be a smooth immersion. The connection $\widehat{\nabla}$ induced on i^*TM by a torsion-free affine connection $\widehat{\nabla}$ on M can be split using a **transversal**, a subbundle $W \subset i^*TM$ transverse to the subbundle identified with $T\Sigma$. Namely, the connection ∇ **induced on** Σ **by** $\widehat{\nabla}$ **and** W is defined by setting $Ti(\nabla_X Y)$ equal to the image of $\widehat{\nabla}_X Ti(Y)$ under the projection of i^*TM onto $Ti(T\Sigma)$ along W, and the **connection** $\bar{\nabla}$ **induced on** W **by** $\widehat{\nabla}$ is defined by setting $\bar{\nabla}_X s$ equal to the projection onto W along $Ti(T\Sigma)$ of $\bar{\nabla}_X s$ for $s \in \Gamma(W)$. Any such transversal W is identified in an obvious way with the normal bundle of Σ, and the connection induced on W is obtained by transporting, via this identification, the connection induced on the normal bundle. On the other hand, the connection induced on Σ depends on the choice of transversal in a fundamental way.

In general a torsion-free affine connection $\widehat{\nabla}$ does not determine a transversal W in any canonical manner, but in many special cases of interest it does, as will be discussed later. First, there is supposed fixed a transverse subbundle W, and there are derived the basic identities relating $\widehat{\nabla}$ and the induced connection ∇.

Let $\widehat{\nabla}$ be a torsion-free affine connection on M and let $i : \Sigma \to M$ be a hypersurface immersion inducing the connection ∇ on M. A (local) transverse vector field $N \in \Gamma(U, W)$ induces an identification of the second fundamental form $\Pi^{\widehat{\nabla}}(i)$ with a tensor $h \in \Gamma(S^2(T^*\Sigma))$, the **shape operator** $S \in \text{End}(T\Sigma)$, and a 1-form $\tau \in \Gamma(T^*\Sigma)$, all defined by

$$\widehat{\nabla}_X Ti(Y) = Ti(\nabla_X Y) + h(X,Y)N, \qquad \widehat{\nabla}_X N = -Ti(S(X)) + \tau(X)N. \tag{6.3}$$

in which, as usual, formally there is required interpretation in terms of vector fields defined near $i(\Sigma)$. If N is replaced by $\tilde{N} = f(N + \tilde{X})$ with $f \neq 0$ and $\tilde{X} = Ti(X)$ along $i(\Sigma)$, then ∇, h, S, and τ are replaced by $\tilde{\nabla}$, $\tilde{h}$, $\tilde{S}$, and $\tilde{\tau}$ satisfying

$$\begin{aligned} &\tilde{h}_{ij} = f^{-1}h_{ij}, \qquad \tilde{S}_i{}^j = f\left(S_i{}^j - \nabla_i X^j + \tau_i X^j + h_{ip}X^pX^j\right), \\ &\tilde{\nabla} - \nabla = -h_{ij}X^k, \qquad \tilde{\tau}_i = \tau_i + d\log|f|_i + h_{ip}X^p. \end{aligned} \tag{6.4}$$

The conformal structure induced by the co-orientation consistent with N is denoted $[h]$. Via the trivialization of the normal bundle given by N the induced connection on $W \simeq \nu\Sigma$ determines a one-form on M, and this one-form is τ.

Definition 6.1. *The* ***mean curvature*** *of* Σ *with respect to* $\widehat{\nabla}$ *and* N *is* $S_p{}^p$.

Write $\hat{R}_{IJK}{}^L$ for the curvature tensor of $\widehat{\nabla}$. Fix a transversal $N \in \Gamma(U, W)$. Letting $P_{T\Sigma W}$ and $P_{WT\Sigma}$ denote the projections of i^*TM onto $T\Sigma$ along W and onto W along $T\Sigma$, define on Σ tensors $\hat{R}_{ijk}{}^l$, $\hat{R}_{ijk}{}^\infty$, $\hat{R}_{ij\infty}{}^k$, and $\hat{R}_{ij\infty}{}^\infty$ (the ∞'s should be regarded as dummy place-holders or as abstract indices on the tensor powers of the one-dimensional normal bundle) by $P_{WT\Sigma}(\hat{R}(X,Y)Z)^l = X^iY^jZ^k\hat{R}_{ijk}{}^l$, $P_{T\Sigma W}(\hat{R}(X,Y)Z) = X^iY^jZ^k\hat{R}_{ijk}{}^\infty N$, $P_{WT\Sigma}(\hat{R}(X,Y)N)^l = X^iY^j\hat{R}_{ij\infty}{}^l$, and $P_{T\Sigma W}(\hat{R}(X,Y)N) = X^iY^j\hat{R}_{ij\infty}{}^\infty N$. Then

$$\hat{R}_{ijk}{}^l = R_{ijk}{}^l - 2S_{[i}{}^lh_{j]k}, \qquad \hat{R}_{ijk}{}^\infty = 2\nabla_{[i}h_{j]k} + 2\tau_{[i}h_{j]k}, \tag{6.5}$$

$$\hat{R}_{ij\infty}{}^k = -2\nabla_{[i}S_{j]}{}^k + 2\tau_{[i}S_{j]}{}^k, \qquad \hat{R}_{ij\infty}{}^\infty = d\tau_{ij} + 2S_{[i}{}^ph_{j]p}. \tag{6.6}$$

The formulas (6.5) and (6.6) are most useful when $\widehat{\nabla}$ is flat, or has some special form (for example, it is the Levi-Civita connection of a metric of constant curvature).

Now suppose that $\widehat{\nabla}$ is the Levi-Civita connection of a Riemannian metric G_{IJ} on M. Let $g_{ij} = i^*(G)_{ij}$ be the Riemannian metric induced on Σ via pullback, and let D be its Levi-Civita connection. It is straightforward to check that if N is chosen to have G-norm 1 and to be orthogonal to Σ, then the connection induced on Σ by $\widehat{\nabla}$ and N equals D. Moreover, in this case $\tau_i = 0$ and $S_i{}^pg_{pj} = h_{ij}$, so that the representative of the second fundamental form and the shape operator determined by N are related via the induced metric.

In this case the mean curvature is also the metric trace of the second fundamental form, for $g^{ij}h_{ij} = S_p{}^p$. Since $S_i{}^j$ is symmetric with respect to g_{ij}, it is diagonalizable. Its eigenvalues are called the **principal curvatures** of Σ.

7. Distance function and Laplacian comparison theorem

The Laplacian comparison theorem says that in a complete Riemannian manifold with Ricci curvature bounded from below by a constant the Laplacian of the distance from a fixed point is bounded from above by the Laplacian of the distance from a fixed point in the constant curvature space corresponding to the given Ricci curvature bound. The Laplacian of the distance from a fixed point can be interpreted as the mean curvature of the equidistant sphere around this point. In more negatively curved spaces geodesics spread apart faster than in less negatively curved spaces, and so intuitively, the mean curvature of equidistant spheres is greater in more negatively curved spaces than it is in less negatively curved spaces. The spreading of geodesics is described in terms of Jacobi fields, and the comparison theorems (of all sorts) are deduced from comparison theorems for ordinary differential equations associated with Jacobi fields. The same ideas can be used to obtain volume comparison theorems. The mean curvature of an equidistant sphere is the infinitesimal distortion of the Riemannian volume form, and via integration this yields comparison results for volume. However, for reasons of space these developments are not pursued here, although they are directly relevant to the topics considered here.

7.1. **Jacobi fields.** A *Jacobi field* along a geodesic segment $\gamma : [0, l] \to M$ is a vector field X defined along $\gamma([0, l])$ and satisfying the differential equation

$$\ddot{X} = D_{\dot{\gamma}} D_{\dot{\gamma}} X = -R(X, \dot{\gamma})\dot{\gamma}. \tag{7.1}$$

By (4.2), a vector field J restricts to a Jacobi field along a geodesic γ if and only if $\mathfrak{L}_X \nabla$ vanishes along γ.

Every Jacobi field along a given geodesic arises as the variation field of a one-parameter family of variations of the geodesic through geodesics.

Lemma 7.1. *Every Jacobi field, X, along the geodesic γ has the form $X = Y + (a + tb)\dot{\gamma}$ where $a, b \in \mathbb{R}$ and $g(Y, \dot{\gamma}) = 0$.*

Proof. It suffices to observe that $g(X, \dot{\gamma})$ solves the second order ordinary differential equation $\ddot{u} = 0$:

$$\tfrac{d^2}{dt^2} g(X, \dot{\gamma}) = \tfrac{d}{dt} g(\dot{X}, \dot{\gamma}) = g(\ddot{X}, \dot{\gamma}) = g(R(X, \dot{\gamma})\dot{\gamma}, \dot{\gamma}) = 0. \tag{7.2}$$

□

Lemma 7.2. *If X and Y are Jacobi fields along the geodesic γ then $g(X, \dot{Y}) - g(\dot{X}, Y)$ is constant along γ.*

Proof. Differentiation yields

$$\begin{aligned} \tfrac{d}{dt}\Big(g(X, \dot{Y}) - g(\dot{X}, Y)\Big) &= g(X, \ddot{Y}) - g(\ddot{X}, Y) = g(R(X, \dot{\gamma})\dot{\gamma}, Y) - g(R(Y, \dot{\gamma})\dot{\gamma}, X) \\ &= g(R(\dot{\gamma}, X)Y, \dot{\gamma}) + g(R(Y, \dot{\gamma})X, \dot{\gamma}) = -g(R(X, Y)\dot{\gamma}, \dot{\gamma}) = 0, \end{aligned} \tag{7.3}$$

the penultimate equality by the algebraic Bianchi identity. □

7.2. **Conjugate points.** Let (M, g) be a Riemannian manifold with Levi-Civita connection D.

The point $q \in M$ is **conjugate to** $p \in M$ if it is a singular value of the exponential map $\exp_p M : T_pM \to M$. That is, there is $v \in T_pM$ such that $q = \exp_p(v)$ and $T\exp_p(v) : T_vT_pM \to T_qM$ is singular. In this case one says that q is conjugate to p **along the geodesic** $\gamma(t) = \exp_p(tv)$. More precisely, the conjugate point $q = \exp_p(v)$ of p has **order** k if $\dim\ker T\exp_p(v) = k$.

For the proofs of the following basic facts see [12].

Lemma 7.3. *A point q is conjugate to p along a geodesic γ if and only if there is a Jacobi field along the geodesic that vanishes only at p and q. Consequently, q is conjugate to p along γ if and only if p is conjugate to q along the time reversal of γ.*

Lemma 7.4. *No geodesic emanating from p is minimizing past the first conjugate point along the geodesic.*

The proof requires some version of the second variation formula.

7.3. **Cut locus and differentiability of the distance function.** In this section (M, g) is an n-dimensional Riemannian manifold and g is assumed complete. While something can be said about the incomplete case, and it is interesting to study this case (for example, see section 3 of [2] and [20, 40] for results about Ricci flow on not necessarily complete Riemannian manifolds), here it would only distract with technical details.

In a Riemannian manifold, it follows immediately from the triangle inequality that the distance $r(q) = d(p, q)$ from a fixed point is a 1-Lipschitz function:

$$|r(q_1) - r(q_2)| = |d(p, q_1 - d(p, q_2)| \leq d(q_1, q_2). \tag{7.4}$$

By the Rademacher Theorem, a Lipschitz function is differentiable almost everywhere, and where r is differentiable, there holds $|dr|^2 = 1$ (see [18]). In practice, it is important to have more precise information about where r is differentiable. The answer is that it is differentiable away from p and the cut locus of p.

For $v \in T_pM$ such that $|v| = 1$, let $\gamma_v : [0, \infty) \to M$ be the geodesic with $\gamma_v(0) = p$ and $\dot{\gamma}_v(0) = v$. Define $c(v) = \sup\{t > 0 : d(p, \gamma_v(t)) = t\}$. Because the exponential map $\exp_p$ is injective on a sufficiently small neighborhood of the origin in T_pM, $c(v) > 0$. If $c(v) < \infty$ then, for any $h > 0$,

$$\begin{aligned} d(\gamma_v(c(v) + h), \gamma_v(0)) &\leq d(\gamma_v(c(v) + h), \gamma_v(c(v))) + d(\gamma_v(c(v))), \gamma_v(0)) \\ &< d(\gamma_v(c(v) + h), \gamma_v(c(v))) + c(v) \leq c(v) + h, \end{aligned} \tag{7.5}$$

so $c(v)$ is the last value of t for which γ_v is minimizing on $[0, t]$. Alternatively, consider the set $S(p, v) = \{t : d(\gamma_v(t), p) = t\} \subset [0, \infty)$. Suppose $r \in [0, \infty) \setminus S(p, v)$. Then (7.5) shows $r + h \in [0, \infty) \setminus S(p, v)$, so that $S(p, v) = [0, c(v))$ has the form $[0, \infty)$ or $[0, r)$ for some $r > 0$. If $c(v) < \infty$, $\gamma_v(c(v))$ is a **cut point of γ_v with respect to** p. The **cut locus** $\mathsf{Cut}(p)$ **of** p is the set of all cut points with respect to p of geodesics emanating from p.

Lemma 7.5. *Let (M, g) be a complete Riemannian manifold. If $q = \gamma(T)$ is a cut point of $p = \gamma(0)$ along the geodesic $\gamma : [0, \infty) \to M$, then at least one, and possibly both, of the following statements holds:*

1. *The first conjugate point of p along γ is q.*
2. *There exist at least two minimizing geodesics from p to q.*

Proof. See Lemma 5.2 in [12] and Theorem 4.2 in [26]. □

Lemma 7.5 means that associated to each $p \in M$ is a partition of M into two sets, namely $\mathsf{Cut}(p) \cup \{p\}$ and $\bar{M} = M \setminus \{\mathsf{Cut}(p) \cup \{p\}\}$. Concretely, $\bar{M}$ consists of those points q such that there exists a unique minimizing geodesic from p to q, that moreover has no conjugate points; its complement consists of those q that can be joined to p either by more than one minimizing geodesic, or by at least one minimizing geodesic along which p and q are conjugate.

Corollary 7.6. *If q is a cut point of p along the geodesic $\gamma(t)$ then p is a cut point of q along the geodesic $\gamma(-t)$.*

The conclusion of Corollary 7.6 could be called *cut point reciprocity*.

Lemma 7.7. *The function c is a continuous function on an open subset of the unit sphere in T_pM. Consequently $\mathsf{Cut}(p)$ is a closed subset of M.*

Proof. See Proposition 5.4 in [12]. □

Theorem 7.8. *Let (M, g) be a Riemannian manifold. The function $r = d(p, \cdot\,)$ is C^∞ on $M(p) = M \setminus (\mathsf{Cut}(p) \cup \{p\})$. Moreover, if $q \in M(p)$ and γ is the unique minimizing geodesic from p to q, then $\operatorname{grad} r_q = \dot\gamma(r(q))$.*

Proof. The following proof is completely standard, but I do not know a good reference.

For $q \in M(p)$ let $X(q) \in T_pM$ be the unique vector such that $\exp_p(d(p,q)X(q)) = q$ and let $U(p) = \{d(p,q)X(q) \in T_pM : q \in M(p)\} \subset T_pM\setminus\{0\}$. By definition of $U(p)$, $\exp_p : U(p) \to M(p)$ is an injective smooth map. Since q is not conjugate to p, $\exp_p$ is not singular at $d(p,q)X(q)$, and so, by the inverse function theorem, $\exp_p$ is a diffeomorphism on some neighborhood of $d(p,q)X(q)$ in $U(p)$. Hence $\exp_p : U(p) \to M(p)$ is a local diffeomorphism; since it is injective, it is a diffeomorphism. Hence $\exp_p^{-1} : M(p) \to U(p)$ is a smooth map. Since $r(q) = d(p,q) = |\exp_p^{-1}(q)|$, this shows that r is smooth on $M(p)$.

Let $q \in M(p)$. Let γ be the unique minimizing geodesic from p to q. Let $X \in T_qM$. Let $\tau : (-\epsilon, \epsilon) \to M(p)$ be a smooth curve such that $\tau(0) = q$ and $\dot\tau(0) = X$. For each $s \in (-\epsilon, \epsilon)$ there is a unique minimizing geodesic $\sigma_s(t)$ from p to $\tau(s)$. This defines a variation of γ through geodesics, and X is the value at q of the corresponding variation field along γ. By the first variation formula (see [12])

$$g_q(\operatorname{grad} r, X) = dr_q(X) = \tfrac{d}{ds}|_{s=0} r(\tau(s)) = \tfrac{d}{ds}|_{t=0} \ell(\sigma_s) = g(X, \dot\gamma(d(p,q))). \tag{7.6}$$

This shows $\operatorname{grad} r = \dot\gamma(r(q))$. □

Remark 7.9. The second statement means that at a point off of the cut locus of p the gradient of the distance from p is the vector tangent to the unique minimizing geodesic from p.

Remark 7.10. The proof of Theorem 7.8 shows that off of the cut locus, r is as smooth as the exponential map. Precisely, if the metric g is C^{k+1}, then the exponential map is C^k, so r is C^k off of the cut locus of p.

Recall the definition of a barrier function, Definition 5.1. Lemma 7.11 is due to Calabi in the proof of Theorem 3 in [8]. It will be needed in the proof of the Laplacian comparison theorem.

Lemma 7.11 (Lemma 7.5 of J. Cheeger's [11])**.** *Let $q \in \mathsf{Cut}(p)$ and let γ be a minimizing geodesic such that $\gamma(0) = p$ and $\gamma(l) = q$ (so $l = d(p,q)$). Then, for all $0 < \epsilon < l$, the function $r_{q,\epsilon}(x) = d(\gamma(\epsilon), x) + \epsilon$ is an upper barrier at q for the function $r(x) = d(p,x)$.*

Proof. By the triangle inequality,

$$r_{q,\epsilon}(x) = d(\gamma(\epsilon), x) + \epsilon = d(\gamma(\epsilon), x) + d(\gamma(\epsilon), \gamma(0)) \geq d(x, \gamma(0)) = d(x,p) = r(p), \tag{7.7}$$

so $r_{q,\epsilon} \geq r$. Since, by Corollary 7.6, p is a cut point of q along the time-reversed geodesic $\sigma(t) = \gamma(l - t)$, where $l = d(p,q)$, this geodesic is minimizing on $[0, l-\epsilon]$, so $l - \epsilon = d(\sigma(l-\epsilon), \sigma(0)) = d(\gamma(\epsilon), q)$. Hence $r_{p,\epsilon}(q) = d(\gamma(\epsilon), \gamma(l)) + \epsilon = l = r(q)$. Since $q \notin \mathsf{Cut}(\gamma(\epsilon))$, by Theorem 7.8 there is an open neighborhood of q contained in the complement of $\mathsf{Cut}(\gamma(\epsilon))$ on which $r_{q,\epsilon}$ is smooth. □

7.4. **Comparison theorem for scalar Ricatti equation.** Geometric comparison theorems are based on comparison theorem for ordinary differential equations.

A Ricatti equation is a first order ordinary differential equation of the form $\dot u = f(u)$ where f is a quadratic polynomial in u.

A useful survey of comparison theorems in Riemannian geometry based on comparison results for Ricatti equations is H. Karcher's [23]. A useful reference discussing comparison results for matrix

Ricatti equations, in the context of Riemannian comparison theorems, is [17]. The reason for the appearance of Ricatti equation, in particular, in Riemannian geometry will be made apparent in Section 7.5.

Define a pair of functions depending on a parameter $\kappa \in \mathbb{R}$:

$$\mathsf{sn}_\kappa(t) = \begin{cases} \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa} t) & \textit{if } \kappa > 0, \\ t & \text{if } \kappa = 0, \\ \frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa} t) & \text{if } \kappa < 0. \end{cases} \tag{7.8}$$

$$\mathsf{ct}_\kappa(t) = \tfrac{d}{dt} \log \mathsf{sn}_\kappa = \begin{cases} \sqrt{\kappa} \cot(\sqrt{\kappa} t) & \textit{if } \kappa > 0, \\ t^{-1} & \text{if } \kappa = 0, \\ \sqrt{-\kappa} \coth(\sqrt{-\kappa} t) & \text{if } \kappa < 0. \end{cases} \tag{7.9}$$

Note that all these functions have the same limiting behavior to first order as $t \to 0$, independent of the value of κ. The function $\mathsf{ct}_\kappa(t)$ solves the Ricatti differential equation

$$\dot{u} + u^2 + \kappa = 0, \qquad u(t) \sim \tfrac{1}{t} \text{ when } t \to 0. \tag{7.10}$$

Lemma 7.12. *Let $u \in C^1((0, R))$ for some $R > 0$ and suppose that, for some $\kappa \in \mathbb{R}$, u satisfies*

$$\dot{u} + u^2 + \kappa \leq 0, \tag{7.11}$$

and $\lim_{t \downarrow 0} u(t) = +\infty$. Then $u(t) \leq \mathsf{ct}_\kappa(t)$ for $t \in (0, R)$.

Proof. This proof follows the proof of Corollary 1.6.2 given in [23].

Let $v(t)$ solve $\dot{v} + v^2 + \kappa = 0$ on some interval $(a, b) \subset (0, \infty)$. For a fixed $t_0 \in (a, b)$, define

$$w(t) = (u(t) - v(t)) \exp\left(\int_{t_0}^t (u(s) + v(s))\, ds\right), \tag{7.12}$$

for $t \in (a, b)$. Then, for $t \in (a, b)$,

$$\dot{w} = \left(\dot{u} - \dot{v} + u^2 - v^2\right) \exp\left(\int_{t_0}^t (u(s) + v(s))\, ds\right) \leq 0. \tag{7.13}$$

Hence w is nonincreasing on (a, b). Consequently, if there is $r_0 \in (a, b)$ such that $u(r_0) \geq v(r_0)$, then $w(r_0) \geq 0$, so $w(t) \geq 0$ on (a, r_0).

Suppose there is $r_0 \in (0, R)$ such that $u(r_0) > \mathsf{ct}_\kappa(r_0)$. Then there is $\epsilon \in (0, r_0)$ such that $u(r_0) \geq \mathsf{ct}_\kappa(r_0 - \epsilon)$. Taking $v(t) = \mathsf{ct}_\kappa(t - \epsilon)$ and applying the claims of the preceeding paragraph on the interval (ϵ, R) yields that $u(t) \geq \mathsf{ct}_\kappa(t - \epsilon)$ on (ϵ, r_0). However, $\lim_{t \downarrow \epsilon} \mathsf{ct}_\kappa(t - \epsilon) = +\infty$ so this contradicts that $u(\epsilon)$ is finite. Hence it must be $u(t) \leq \mathsf{ct}_\kappa(t)$ for all $t \in (0, R)$. □

Alternative proof of Lemma 7.12. In most references one finds some variant of the following proof. Since $\lim_{t \downarrow 0} u(t) = +\infty$, there is some maximal subinterval $(0, T) \subset (0, R)$ on which u is positive. Precisely, let $T = \sup\{t \in (0, R) : f(s) > 0 \text{ if } s \in (0, t)\}$. Then, for $t \in (0, T)$, $w(t) = 1/u(t)$ satisfies

$$-\dot{w} + 1 + \kappa w^2 = u^{-2}(\dot{u} + u^2 + \kappa) \leq 0. \tag{7.14}$$

When κ is nonnegative, there is no further issue, but when κ is negative, it could be that $1 + \kappa w(t)^2 = 0$ for some $t \in (0, T)$ (in many books this point is ignored). Since $\lim_{t \downarrow 0} w(t) = 0$, there is some, perhaps smaller, maximal subinterval $(0, S) \subset (0, T)$ on which $1 + \kappa w^2$ is positive. Precisely, let $S = \sup\{t \in (0, T) : 1 + \kappa w(s)^2 > 0 \text{ if } s \in (0, t)\}$. For $t \in (0, S)$ there holds

$$t = \int_0^t ds \leq \int_0^t \tfrac{\dot{w}(s)}{1 + \kappa w(s)^2}\, ds = \int_0^t \tfrac{d}{ds} \mathsf{at}_\kappa(w(s))\, ds = \mathsf{at}_\kappa(w(t)), \tag{7.15}$$

where

$$\mathsf{at}_\kappa(t) = \begin{cases} \frac{1}{\sqrt{\kappa}} \arctan(\sqrt{\kappa} t) & \text{if } \kappa > 0, \\ t & \text{if } \kappa = 0, \\ \frac{1}{\sqrt{-\kappa}} \operatorname{arctanh}(\sqrt{-\kappa} t) & \text{if } \kappa < 0, \end{cases} \tag{7.16}$$

and there has been used $\lim_{t \downarrow 0} \mathsf{at}_\kappa(t) = 0$. (Many presentations of this variant of the proof appear to assume that the integral in (7.15) converges.) Consequently, for $t \in (0, S)$, there holds $t \leq \mathsf{at}_\kappa(w(t))$, and this implies that

$$u(t) \leq \mathsf{ct}_k(t) \tag{7.17}$$

for $t \in (0, S)$. Since this implies that $1 + \kappa w^2$ is positive at S, it must be that $S = T$. Likewise, since $w(t) \geq 1/\mathsf{ct}_k(t) > 0$, this then implies that w, and so also u, is positive on $(0, T)$, and hence that $T = R$. □

7.5. **Comparison theorems for the distance function.** The Laplacian comparison theorem shows that, for a complete Riemannian manifold with Ricci curvature bounded from below, the Laplacian of the distance from a fixed point p is bounded from above by the Laplacian of the distance from a fixed point in a constant curvature model space corresponding to the given bound. The statement makes sense in a strong sense away from the cut locus of p, where the distance from p is differentiable, but has to be understood in a weak sense to have sense globally. Although the result itself is used in most applications of the maximum principle method, and will be used repeatedly in what follows, the techniques required in its proof are equally important.

The fundamental idea is to interpret the Laplacian of the distance r from p as the mean curvature (with respect to the inner unit normal) of the geodesic sphere of radius r around p.

Lemma 7.13. *Let (M, g) be a complete Riemannian manifold. For $p \in M$ let $r(q) = d(p, q)$ and let $M(p) = M \setminus \{\mathsf{Cut}(p) \cup \{p\}\}$. For $q \in M(p)$:*

1. *The second fundamental form at q of the level set of r passing through q, the geodesic sphere $\partial B(p, d(p, q))$, is the restriction to $T_q \partial B(p, d(p, q))$ of* $\operatorname{Hess} r = Ddr$.
2. *The mean curvature $\mathcal{H}$ at q of $\partial B(p, d(p, q))$ is $\frac{1}{n-1} \Delta r$.*
3. *The shape operator at q, S, of $\partial B(p, d(p, q))$ satisfies the differential equation*

$$D_{\partial_r} S + S^2 + R(\,\cdot\,, \partial_r)\partial_r = 0. \tag{7.18}$$

Proof. Fix $p \in M$ and let $q \in M \setminus \{\mathsf{Cut}(p) \cup \{p\}\}$. For $l = d(p, q)$, let $\gamma : [0, l] \to M$ be a minimizing geodesic such that $\gamma(0) = p$ and $\gamma(l) = q$. Let X be a Jacobi field along γ such that $g(X(0), \dot\gamma(0)) = 0 = g(\dot X(0), \dot\gamma(0))$. Then $g(X, \dot\gamma) = 0 = g(\dot X, \dot\gamma)$ for all $t \in [0, l]$. Write ∂_r for $\operatorname{grad} r$. By Theorem 7.8, at $q \in M(p)$, $\partial_r = \operatorname{grad} r = \dot\gamma$.

By the Gauss lemma (see [12]), the inward unit normal to $\partial B(p, d(p, q))$ at q is $N = -\partial_r$. The shape operator S is defined by $S(X) = -D_X N = D_X \partial_r$ for X tangent to $\partial B(p, d(p, q))$. For any vector fields X and Y,

$$\operatorname{Hess} r(X, Y) = (D_X dr)(Y) = X g(\partial_r, Y) - g(\partial_r, D_X Y) = g(D_X \partial_r, Y) = g(S(X), Y). \tag{7.19}$$

In particular, if X and Y are tangent to $\partial B(p, d(p, q))$ at q, then the second fundamental form Π satisfies $\Pi_q(X, Y) = g_q(S(X), Y) = (D_X dr)(Y)$. Moreover, $\operatorname{Hess} r(X, N) = -g(S(X), \partial_r) = -dr(X)$ vanishes at q, because X is tangent to the level set of r through q, and $\operatorname{Hess} r(N, N) = -g(D_{\partial_r} \partial_r, \partial_r) = -\frac{1}{2} \partial_r |\partial_r|^2 = 0$, because $|\partial_r|^2 = 1$. Hence $\operatorname{tr} S = \Delta r$.

Let $\dot\gamma(t)^\perp = \{v \in T_{\gamma(t)} M : g(v, \dot\gamma(t)) = 0\}$. Define $\mathcal{S}(t) \in \operatorname{End}(\dot\gamma(t)^\perp)$ by $\mathcal{S}(t) X(t) = \dot X(t)$. Define $\mathcal{R}(t) \in \operatorname{End}(\dot\gamma^\perp)$ by $\mathcal{R}(t) v = R(v, \dot\gamma(t))\dot\gamma(t)$. Then

$$-\mathcal{R} X = -R(X, \dot\gamma)\dot\gamma = \ddot X = D_{\dot\gamma}(\mathcal{S} X) = (D_{\dot\gamma} \mathcal{S}) X + \mathcal{S} \dot X = (D_{\dot\gamma} \mathcal{S}) X + \mathcal{S}^2 X. \tag{7.20}$$

Hence

$$\dot{\mathcal{S}} + \mathcal{S}^2 + \mathcal{R} = 0, \tag{7.21}$$

where $\dot{\mathcal{S}} = D_{\dot\gamma}\mathcal{S}$. Since the Jacobi field X is the variation field of a variation of γ through geodesics, $\mathcal{S}X = \dot{X} = D_{\partial_r}X = D_X\partial_r = SX$, so $\mathcal{S}(t) = S_{\gamma(t)}$ is the shape operator of $\partial B(p,t)$. Since $\dot\gamma(l) = \partial_r$, (7.21) yields (7.18). □

Lemma 7.14. *The mean curvature of the boundary of a geodesic ball of radius R in a simply connected complete space of constant sectional curvature κ is $\mathsf{ct}_\kappa(R)$.*

Proof. Let the notation be as in the proof of Lemma 7.13. The operator $\mathcal{R}(t)$ is diagonal, equal to κ times the identity. Hence the equation (7.21) with the initial condition that the eigenvalues of $\mathcal{S}(t)$ go to ∞ as $t \to \infty$ admits a unique solution that is diagonal. That is (7.21) reduces to a scalar equation $\dot{s} + s^2 + \kappa = 0$. The solution with the stated initial conditions is ct_κ. Since $\operatorname{tr} S = (n-1)s$, this proves the claim. □

The $\kappa = 0$ case of Theorem 7.15 was proved by E. Calabi as Theorem 1 of [8]. Since the proof for other values of κ is essentially the same, it seems correct to attribute the theorem to Calabi.

Theorem 7.15. *Let (M, g) be a complete Riemannian manifold having Ricci curvature satisying $R_{ij} \geq \kappa(n-1)g_{ij}$ for some constant $\kappa \in \mathbb{R}$. For $p \in M$ let $r(q) = d(p,q)$. On $M \setminus \{\mathsf{Cut}(p) \cup \{p\}\}$ there holds*

$$\Delta r \leq (n-1)\mathsf{ct}_\kappa(r) \tag{7.22}$$

in the strong sense (pointwise). Moreover, (7.22) *holds in the barrier sense on all of M.*

Proof. Let $M(p) = M \setminus \{\mathsf{Cut}(p) \cup \{p\}\}$. Let $q \in M(p)$. For $l = d(p,q)$, let $\gamma : [0,l] \to M$ be a minimizing geodesic such that $\gamma(0) = p$ and $\gamma(l) = q$.

It is claimed that the mean curvature $\mathcal{H}(q)$ of $\partial B(p, d(p,q))$ at q satisfies the differential inequality

$$\partial_r \mathcal{H} + \mathcal{H}^2 + \kappa \leq 0. \tag{7.23}$$

There are given two different derivations of (7.25). The first derivation works directly with the function r. Applying (4.46) with $f = r$ yields

$$\begin{aligned} 0 = \tfrac{1}{2}\Delta|dr|^2 &\geq \tfrac{1}{n-1}(\Delta r)^2 + r^p(\Delta r)_p + r^p r^q R_{pq} \\ &= \tfrac{1}{n-1}(\Delta r)^2 + \partial_r \Delta r + \mathsf{Ric}(\partial_r, \partial_r) \\ &\geq \tfrac{1}{n-1}(\Delta r)^2 + \partial_r \Delta r + \kappa(n-1). \end{aligned} \tag{7.24}$$

By (7.24), along γ the function $u(t) = (\Delta r)(\gamma(t)) = (n-1)\mathcal{H}(\gamma(t))$ satisfies

$$0 \geq \dot{u} + \tfrac{1}{n-1}u^2 + \kappa(n-1) = (n-1)\left(\tfrac{d}{dt}\left(\tfrac{u}{n-1}\right) + \left(\tfrac{u}{n-1}\right)^2 + \kappa\right). \tag{7.25}$$

The second derivation works with the shape operator and the equation (7.18). Let $\mathcal{S}(t)$ be defined as in the proof of Lemma 7.13. From the nonnegativity of $|A - \frac{1}{m}(\operatorname{tr} A)I|^2$ it follows that an endomorphism A of an m-dimensional vector space satisfies $m \operatorname{tr} A^2 \geq (\operatorname{tr} A)^2$. Note that $\operatorname{tr} R(t) = \mathsf{Ric}(\dot\gamma, \dot\gamma)$. Hence, tracing (7.21) yields

$$0 \geq \operatorname{tr}\dot{\mathcal{S}} + \tfrac{1}{n-1}(\operatorname{tr}\mathcal{S})^2 + \mathsf{Ric}(\dot\gamma,\dot\gamma). \tag{7.26}$$

That is, the function $u(t) = \operatorname{tr}\mathcal{S}(t) = (n-1)\mathcal{H}(\gamma(t))$ satisfies

$$\dot{u} + \tfrac{1}{n-1}u^2 + \mathsf{Ric}(\dot\gamma,\dot\gamma) \leq 0. \tag{7.27}$$

If $R_{ij} \geq \kappa(n-1)g_{ij}$ for a constant $\kappa \in \mathbb{R}$, there follows (7.25).

The proof follows from Lemma 7.12 once it is shown that $\lim_{t\downarrow 0} u(t) = +\infty$. This follows from the fact that, as $t \to 0$, the geodesic ball around p of radius t closely approximates the radius t ball around the origin in T_pM. Precisely, on a ball $B(0, r) \subset T_pM$ on which $\exp_p$ is an embedding (there always exists such a ball), the exponential map $\exp_p$ maps radial geodesic segments to geodesic segments emanating from p of the same length (see chapter 1 of [12] for precise statements), and thus maps a small sphere around the origin in T_pM diffeomorphically onto a small geodesic sphere centered on p and having the same radius. Since the norm of the differential of $\exp_p$ is bounded on a small ball around $0 \in T_pM$, the distortion of the mean curvatures of these spheres is bounded, so the mean curvature of $\partial B(p, t)$ tends to $+\infty$ as $t \downarrow 0$. Hence, by Lemma 7.12 and (7.23) of Lemma 7.13, the mean curvature $\mathcal{H}(q) = \frac{1}{n-1}(\Delta r)(q)$ satisfies $\mathcal{H} = \frac{1}{n-1}\Delta r \leq (n-1)\mathsf{ct}_k(r)$.

Let $q \in \mathsf{Cut}(p)$ and let γ be a minimizing geodesic such that $\gamma(0) = p$ and $\gamma(d(p, q)) = q$. Given $\epsilon > 0$, choose $\sigma \in (0, d(p, q))$ such that $(n-1)\mathsf{ct}_\kappa(r-\sigma) \leq (n-1)\mathsf{ct}_k(r) + \epsilon$. By Lemma 7.11, there is an open neighborhood $V_{q,\epsilon}$ on which $r_{q,\epsilon} = d(\gamma(\sigma), q) + \sigma$ is an (smooth) upper barrier for $r = d(p, \cdot\,)$. By (7.22), on $V_{q,\epsilon}$ there holds

$$\Delta r_{q,\epsilon} \leq (n-1)\mathsf{ct}_\kappa(r_{q,\epsilon} - \sigma) \leq (n-1)\mathsf{ct}_\kappa(r-\sigma) \leq (n-1)\mathsf{ct}_k(r) + \epsilon, \tag{7.28}$$

the penultimate inequality because ct_κ is monotone decreasing and $r \leq r_{q,\sigma}$. This shows that (7.22) holds in the barrier sense on the cut locus of p. □

Remark 7.16. By (3.11) with $f = r$,

$$\mathfrak{L}_{\text{grad}\, r}\mathsf{vol}_g = (\Delta r)\mathsf{vol}_g. \tag{7.29}$$

Hence the mean curvature of the sphere equidistant from a point measures the volume distortion along this sphere. This observation underlies the proofs of volume comparison theorems.

Corollary 7.17 was proved by Calabi in [8]; see alo Proposition 7.7 of [11].

Corollary 7.17. *[Corollary of Theorem 7.15] Let (M, g) be a complete Riemannian manifold having Ricci curvature satisying $R_{ij} \geq \kappa(n-1)g_{ij}$ for some constant $\kappa \in \mathbb{R}$. For $p \in M$ let $r(q) = d(p, q)$. If ϕ is twice differentiable monotone increasing function on some interval $[a, b] \subset (0, \infty)$, there holds*

$$\Delta\phi(r) \leq \phi''(r) + (n-1)\mathsf{ct}_\kappa(r)\phi'(r). \tag{7.30}$$

in the barrier sense. If ϕ is twice differentiable monotone decreasing function on some interval $[a, b] \subset (0, \infty)$, there holds

$$\Delta\phi(r) \geq \phi''(r) + (n-1)\mathsf{ct}_\kappa(r)\phi'(r). \tag{7.31}$$

in the barrier sense.

Proof. It clearly suffices to prove one of (7.30) and (7.31). Let $p \in M$. For $q \in M \setminus \{\mathsf{Cut}(p) \cup \{p\}\}$ the claim follows from Theorem 7.15. The proof that (7.30) holds in the barrier sense is that same as the proof that (7.22) holds in the barrier sense. □

8. Growth of functions satisfying a differential inequality

The problem of studying the growth of solutions of an inequality of the form $\Delta u \geq f(u)$ has a long history. Standard references include Keller's article [24] and Osserman's article [32]. Theorem 4 of E. Calabi's [8] extends such results to the context of Riemannian manifolds; it gives a condition guaranteeing that on a complete manifold with nonnegative Ricci curvature a differential inequality of the form $\Delta u \geq f(u)$ has no solution. In [13], S. Y. Cheng and S. T. Yau extended these results to complete Riemannian manifolds having Ricci curvature bounded from below, and obtained an estimate of the supremum of a solution of $\Delta u \geq f(u)$.

Here the focus is on the special case $f(t) = Bt^{1+\sigma} - At$, where A, B, and σ are positive constants. In this case $c = (A/B)^{1/\sigma}$ solves $f(c) = 0$ so solves the inequality $\Delta u \geq f(u)$. The conclusion of

Theorem 8.1 is that, with suitable hypotheses on (M, g), any solution of $\Delta u \geq f(u)$ is bounded from above by the zero $c = (A/B)^{1/\sigma}$ of f. This can be deduced from the general Theorem 8 of [13]. The content of Theorem 8.1 is essentially that of Theorem 5 and its Corollaries 1 and 2 in Section 4 of [15], and the argument is also the same one used by Cheng and Yau to prove Theorem 2 of their [14], which theorem estimates the growth of the norm of the second fundamental form of a complete maximal spacelike hypersurface in Minkowski space. The argument is also quite similar to the original proof of the gradient estimate as Theorem 6 of [13]. Here a full proof is given following Section 4 of [15] (equivalently Section 3 of [14]) and the proof of Yau's gradient estimate for harmonic functions given in I.3 of [37]. The statement of Theorem 8.1 can be found in the form given here, in the more general context of metric measure spaces, as Theorem 3.2 of [19], and, in the context of Hermitian manifolds, as Theorem 4.1 of [41].

It is useful to have a local version of Theorem 8.1. Here such a result is proved as Theorem 8.2 below. This is a strictly stronger result that implies Theorem 8.1 as an immediate corollary. The structure of the proof is essentially the same, only the test function has to be chosen more carefully. The utility of such a local estimate is shown by deducing from it as a corollary the sharp local gradient estimate for harmonic functions.

Theorem 8.1 (Cheng-Yau. [14], [15])**.** *Let (M, g) be a complete n-dimensional Riemannian manifold with Ricci curvature bounded from below by $-\kappa^2(n-1)g_{ij}$ for some real constant $\kappa \geq 0$. Suppose $u \in C^\infty(M)$ is nonnegative and not identically 0 and satisfies*

$$\Delta u \geq B u^{1+\sigma} - Au \tag{8.1}$$

for constants $B > 0$, $\sigma > 0$, and $A \in \mathbb{R}$. Then for any $x \in M$ at which $u(x) \neq 0$, and any $R \in (0, \infty)$, on the open ball $B(x, R)$ of radius R centered at x there holds

$$u^\sigma \leq (R^2 - r^2)^{-2} B^{-1} \left(AR^4 + \tfrac{4\kappa(n-1)}{\sigma} R^3 + \tfrac{4(n+2)\sigma + 16}{\sigma^2} R^2 \right), \tag{8.2}$$

where $r = d(x, \cdot\,)$ is the distance from x. If $n > 2$ and $\sigma > 4/(n-2)$, then for any $x \in M$ at which $u(x) \neq 0$, and any $R \in (0, \infty)$, on the open ball $B(x, R)$ of radius R centered at x there holds

$$u^\sigma \leq (R^2 - r^2)^{-2} B^{-1} \left(AR^4 + \tfrac{4\kappa(n-1)}{\sigma} R^3 + \tfrac{4n}{\sigma} R^2 \right). \tag{8.3}$$

In particular, letting $R \to \infty$:

(1) if $A \leq 0$, then u is identically zero;
(2) if $A > 0$, there holds

$$\sup_M u \leq |A/B|^{1/\sigma}. \tag{8.4}$$

Proof. Let $u \in C^\infty(M)$ and suppose $u \geq 0$ and u is not identically zero. Suppose x chosen so that $u(x) \neq 0$. Fix $R > 0$. Let $\phi : [0, \infty) \to \mathbb{R}$ be a function (to be specified later) satisfying $\phi(0) > 0$, $\phi(t) = 0$ for $t > R$, and $\phi' \leq 0$. Note that these conditions imply $\phi > 0$ on $[0, R)$. In particular it makes sense to write $\psi(t) = \log \phi(t)$ for $t \in [0, R)$. Define $f = \phi(r)u$ where r is the distance from x. By assumption and construction, f is nonnegative and not identically zero on $B(x, R)$. Since r is smooth on the complement of $\{x\} \cup \mathsf{Cut}(x)$, f is smooth on the complement $B(x, R) \setminus \{\{x\} \cup \mathsf{Cut}(x)\}$ in the ball $B(x, R)$, and there hold

$$df = f(d \log u + d \log \phi) = f(d \log u + d\psi), \tag{8.5}$$

$$\Delta f = f\, |d \log u + d\psi|^2 + f(\Delta \log u + \Delta\psi). \tag{8.6}$$

The restriction of f to the boundary $\partial B(x, R)$ is identically 0, and since f is not identically 0 on $B(x, R)$, it must be that its restriction to the closure of $B(x, R)$ attains its maximum at some point $x_0 \in B(x, R)$. First suppose $x_0 \notin \mathsf{Cut}(x)$. The proof in the case $x_0 \in \mathsf{Cut}(x)$ is similar, and will be described at the end. Since $f(x_0) \neq 0$, also $u(x_0) \neq 0$. It follows from (8.5) that at x_0 there holds $d\log u = -d\psi$. At x_0 there holds $0 \geq \Delta f$, and in (8.6) this implies that at x_0 there holds

$$0 \geq f^{-1}\Delta f = |d\log u + d\psi|^2 + Bu^\sigma - A - |d\log u|^2 + \Delta\psi = Bu^\sigma - A - |d\psi|^2 + \Delta\psi. \tag{8.7}$$

Consequently, at x_0, there holds

$$Bu^\sigma \leq A + |d\psi|^2 - \Delta\psi. \tag{8.8}$$

By the Laplacian comparison theorem the assumed lower bound on the Ricci curvature implies that on $M \setminus \mathsf{Cut}(x)$ there holds

$$\Delta r \leq (n-1)(r^{-1} + \kappa). \tag{8.9}$$

Since $\psi' = \phi^{-1}\phi'$ is nonpositive on $[0, R)$, from (8.9) there follows

$$\Delta\psi = \psi''|dr|^2 + \psi'\Delta r \geq \psi'' + (n-1)(r^{-1} + \kappa)\psi'. \tag{8.10}$$

In (8.8) this yields that, at x_0, there holds

$$Bu^\sigma \leq A + (\psi')^2 - \psi'' - (n-1)(r^{-1} + \kappa)\psi' \tag{8.11}$$

Although f assumes its maximum at x_0, this certainly need not be the case for u, so (8.11) does not yield immediately information for u on all of $B(x, R)$. The idea now is to choose ψ so that the right-hand side of (8.11) admits an upper bound independent of x. Multiplying both sides of (8.11) by ϕ^σ yields

$$Bf^\sigma \leq \phi^\sigma \left(A + (\psi')^2 - \psi'' - (n-1)(r^{-1} + \kappa)\psi'\right). \tag{8.12}$$

Since f assumes its maximum on $B(x, R)$ at x_0, if the right-hand side of (8.12) can be bounded independently of x_0, then a bound is obtained on f over the whole ball $B(x, R)$.

Let $\phi(t) = (R^2 - t^2)^\epsilon$ (so that $\psi = \epsilon\log(R^2 - r^2)$ where $\epsilon > 0$ will be chosen at the end. Then

$$\begin{aligned} \psi'(r) &= -\tfrac{2\epsilon r}{R^2 - r^2}, \\ \psi''(r) &= -\tfrac{2\epsilon}{R^2 - r^2} - \tfrac{4\epsilon r^2}{(R^2 - r^2)^2}. \end{aligned} \tag{8.13}$$

In (8.12) there results

$$B(\max_{B(x,R)} f)^\sigma \leq (R^2 - r^2)^{\sigma\epsilon}\left(A + \tfrac{2\epsilon(n + \kappa(n-1)r)}{R^2 - r^2} + \tfrac{4\epsilon(\epsilon+1)r^2}{(R^2 - r^2)^2}\right), \tag{8.14}$$

valid at x_0. Choosing $\epsilon = 2\sigma^{-1}$ yields

$$\begin{aligned} B(\max_{B(x,R)} f)^\sigma &\leq A(R^2 - r^2)^2 + 4\sigma^{-1}(n + \kappa(n-1)r)(R^2 - r^2) + 8\sigma^{-2}(2 + \sigma)r^2 \\ &\leq AR^4 + \tfrac{4\kappa(n-1)}{\sigma}R^3 + \tfrac{4(n+2)\sigma + 16}{\sigma^2}R^2. \end{aligned} \tag{8.15}$$

If $n > 2$ and $\sigma > 4/(n-2)$ then $4 - (n-2)\sigma < 0$, so (8.15) can be replaced by

$$\begin{aligned} B(\max_{B(x,R)} f)^\sigma &\leq A(R^2 - r^2)^2 + 4\sigma^{-1}(n + \kappa(n-1)r)(R^2 - r^2) + 8\sigma^{-2}(2 + \sigma)r^2 \\ &\leq AR^4 + 4\sigma^{-1}\kappa(n-1)R^3 + 4\sigma^{-1}n(R^2 - r^2) + 8\sigma^{-2}(2 + \sigma)r^2 \\ &= AR^4 + 4\sigma^{-1}\kappa(n-1)R^3 + 4\sigma^{-1}nR^2 + 4\sigma^{-2}(4 - (n-2)\sigma)r^2 \\ &\leq AR^4 + 4\sigma^{-1}\kappa(n-1)R^3 + 4\sigma^{-1}nR^2. \end{aligned} \tag{8.16}$$

Since the right-hand side of (8.15) does not depend on x_0 there results

$$B(R^2 - r^2)^2 u^\sigma \leq AR^4 + \tfrac{4\kappa(n-1)}{\sigma}R^3 + \tfrac{4(n+2)\sigma + 16}{\sigma^2}R^2, \tag{8.17}$$

on $B(x, R)$. Dividing both sides by $B(R^2 - r^2)^2$ yields (8.2). If $n > 2$ and $\sigma > 4(n-2)$, since the right-hand side of (8.16) does not depend on x_0 there results

$$B(R^2 - r^2)^2 u^\sigma \leq AR^4 + \tfrac{4\kappa(n-1)}{\sigma} R^3 + \tfrac{4n}{\sigma} R^2, \tag{8.18}$$

on $B(x, R)$. Dividing both sides by $B(R^2 - r^2)^2$ yields (8.3).

For the proof in the case $x_0 \in \mathsf{Cut}(x)$ the argument is modified as in the proof of Yau's gradient estimate on page 21 in Section I.3 of [37]. This is recalled here now. By the assumption that $x_0 \in \mathsf{Cut}(x)$ there is a minimizing geodesic joining x to x_0 the image σ of which necessarily lies in $B(x, R)$ (because no point of σ is farther from x than is x_0). Let $\bar{x}$ be a point on σ lying strictly between x and x_0 at some distance $\epsilon > 0$ from x. Since σ is minimizing, no point of the interior of σ can be conjugate to $\bar{x}$. Were x or x_0 conjugate to $\bar{x}$ then $\bar{x}$ would be in its cut locus, which it is not because $x_0 \in \mathsf{Cut}(x)$ (and so $x \in \mathsf{Cut}(x_0)$). Thus no point of σ is a conjugate point of $\bar{x}$ and hence there is some $\delta > 0$ for which there is an open δ neighborhood $N \subset B(x, a)$ of σ containing no conjugate point of $\bar{x}$. Let $\bar{r} = d(\bar{x}, \cdot\,)$. By the triangle inequality, $\bar{r} + \epsilon \geq r$. On the other hand $\bar{r}(x_0) + \epsilon = r(x_0)$. Define $\bar{f} = (R^2 - (\bar{r} + \epsilon)^2)^\epsilon u$. Then $\bar{f} \leq f$ on N and $\bar{f}(x_0) = f(x_0)$, so $\bar{f}$ attains its maximum value on N at x_0. As $\bar{r}$ is smooth near x_0 the preceeding argument goes through with $\bar{f}$ in place of f, and letting $\epsilon \to 0$ at the end yields (8.2).

Finally, if $A \leq 0$ then letting $R \to \infty$ in (8.2) yields $0 \leq \sup_M u \leq 0$, so u must be identically zero. $\square$

Theorem 8.2 is a local version of Theorem 8.1. The proof is the same, except the test function is more carefully chosen.

Theorem 8.2. *Let (M, g) be a complete n-dimensional Riemannian manifold. Suppose that, for $R > 0$, the Ricci curvature is bounded from below on the open geodesic ball $B(p, 2R)$ by $-\kappa(n-1)g_{ij}$ for some real constant $\kappa \geq 0$. Suppose $u \in C^\infty(B(p, 2R))$ is nonnegative and not identically 0 and suppose there are constants $B > 0$, $\sigma > 0$, and $A \in \mathbb{R}$ such that*

$$\Delta u \geq Bu^{1+\sigma} - Au \tag{8.19}$$

on $B(p, 2R)$. There is a constant $C = C(n, \kappa) > 0$, depending only on n and κ, such that

$$\sup_{B(p,R)} u^\sigma \leq B^{-1}(A + C\sigma^{-1}R^{-1}). \tag{8.20}$$

In particular, letting $R \to \infty$:

1. *if $A \leq 0$, then u is identically zero;*
2. *if $A > 0$, there holds*

$$\sup_M u \leq (A/B)^{1/\sigma}. \tag{8.21}$$

Proof. Let $r(q) = d(p, q)$ be the distance from p. Let $\phi \in C^\infty([0, \infty)$ be a nonnegative function that satisfies $\phi(x) = 1$ on $[0, R]$, $\phi(x) = 0$ on $[2R, \infty)$, and $\dot{\phi} \leq 0$ on $[0, \infty)$. The precise form of ϕ will be specified later. These conditions imply ϕ is positive on $[0, 2R)$, so it makes sense to write $\psi(t) = \log \phi(t)$ for $t \in [0, 2R)$. Define $f = \phi(r)u$. By assumption and construction, f is nonnegative and not identically zero on $B(p, 2R)$. Since r is smooth on $B(p, 2R) \setminus \{\{p\} \cup \mathsf{Cut}(p)\}$, f is smooth on the complement $B(p, 2R) \setminus \{\{p\} \cup \mathsf{Cut}(p)\}$, and there hold

$$df = f(d \log u + d \log \phi) = f(d \log u + d\psi), \tag{8.22}$$

$$\Delta f = f\, |d \log u + d\psi|^2 + f(\Delta \log u + \Delta\psi). \tag{8.23}$$

The restriction of f to the boundary $\partial B(p, 2R)$ is identically 0, and since f is not identically 0 on $B(p, 2R)$, its restriction to the closure of $B(p, 2R)$ attains its maximum value at some point $x_0 \in B(p, 2R)$. First suppose $x_0 \notin \{\{p\} \cup \mathsf{Cut}(p)\}$. The proof in the case $x_0 \in \{\{p\} \cup \mathsf{Cut}(p)\}$ is similar, and will be described at the end. Since $f(x_0) \neq 0$, also $u(x_0) \neq 0$. It follows from (8.22) that at x_0 there holds $d\log u = -d\psi$. At x_0 there holds $0 \geq \Delta f$, and in (8.23) this implies that at x_0 there holds

$$0 \geq f^{-1}\Delta f = |d\log u + d\psi|^2 + Bu^\sigma - A - |d\log u|^2 + \Delta\psi = Bu^\sigma - A - |d\psi|^2 + \Delta\psi. \tag{8.24}$$

Consequently, at x_0, there holds

$$Bu^\sigma \leq A + |d\psi|^2 - \Delta\psi. \tag{8.25}$$

By the Laplacian comparison theorem the assumed lower bound on the Ricci curvature implies that on $M \setminus \{\{p\} \cup \mathsf{Cut}(p)\}$ there holds

$$\Delta r \leq (n-1)(r^{-1} + \sqrt{\kappa}). \tag{8.26}$$

Since $\psi' = \phi^{-1}\phi'$ is nonpositive on $[0, 2R)$, from (8.26) there follows

$$\Delta\psi = \psi''|dr|^2 + \psi'\Delta r \geq \psi'' + (n-1)(r^{-1} + \sqrt{\kappa})\psi'. \tag{8.27}$$

In (8.25) this yields that, at x_0, there holds

$$Bu^\sigma \leq A + (\psi')^2 - \psi'' - (n-1)(r^{-1} + \sqrt{\kappa})\psi' \tag{8.28}$$

Although f assumes its maximum at x_0, this certainly need not be the case for u, so (8.28) does not yield immediately information for u on all of $B(p, 2R)$. The idea now is to choose ψ so that the right-hand side of (8.28) admits an upper bound independent of x_0. Multiplying both sides of (8.11) by ϕ^σ yields

$$Bf^\sigma \leq \phi^\sigma\left(A + (\psi')^2 - \psi'' - (n-1)(r^{-1} + \sqrt{\kappa})\psi'\right). \tag{8.29}$$

Since f assumes its maximum on $B(p, 2R)$ at x_0, if the right-hand side of (8.29) can be bounded independently of x_0, then a bound is obtained on f. Such a bound will be obtained on the smaller ball $B(p, R)$.

Define the function ϕ as follows. The function

$$\tau(t) = \begin{cases} 0 & \text{if } t = 0, \\ \left(1 + \exp(\frac{2}{t} + \frac{2}{t-1})\right)^{-1} & \text{if } 0 < t < 1, \\ 1 & \text{if } t \geq 1, \end{cases} \tag{8.30}$$

is infinitely differentiable on $[0, \infty)$ with bounded first and second derivatives. Observe that $\tau(t) + \tau(1-t) = 1$. The derivative of τ is

$$\dot{\tau}(t) = \begin{cases} 0 & \text{if } t = 0, \\ \frac{\left(\frac{2}{t^2} + \frac{2}{(t-1)^2}\right)\exp(\frac{2}{t} + \frac{2}{t-1})}{\left(1 + \exp(\frac{2}{t} + \frac{2}{t-1})\right)^2} & \text{if } 0 < t < 1, \\ 0 & \text{if } t \geq 1, \end{cases} \tag{8.31}$$

As t approaches 0 from above or 1 from below, the derivative $\tau'(t)$ tends to 0, from which it follows that $\tau'(t)$ is bounded on $[0, \infty)$. That τ'' is bounded can be checked similarly. Note also that τ' is positive on $(0, 1)$.

Define $\phi(t) = \tau(2 - t/R)^\alpha$, where $\alpha > 0$ is a constant that will be specified later. Thus

$$\phi(t) = \begin{cases} 1 & \text{if } t \leq R, \\ \left(1 + \exp\left(\frac{2}{2 - \frac{t}{R}} + \frac{2}{1 - \frac{t}{R}}\right)\right)^{-\alpha} & \text{if } R < t < 2R, \\ 0 & \text{if } t \geq 2R, \end{cases} \tag{8.32}$$

Then

$$\dot{\phi} = -\alpha R^{-1}\tau^{\alpha-1}\dot{\tau}, \qquad \dot{\psi} = -\alpha R^{-1}\tau^{-1}\dot{\tau}, \tag{8.33}$$

$$\ddot{\phi} = \alpha(\alpha-1)R^{-2}\tau^{\alpha-2}\dot{\tau}^2 + \alpha R^{-2}\tau^{\alpha-1}\ddot{\tau}, \qquad \ddot{\psi} = \alpha R^{-2}(\tau^{-1}\ddot{\tau} - \tau^{-2}\dot{\tau}^2). \tag{8.34}$$

Hence

$$\begin{aligned} &\phi^\sigma\left((\psi')^2 - \psi'' - (n-1)(r^{-1} + \sqrt{\kappa})\psi'\right) \\ &\quad = \alpha R^{-1}\tau^{\sigma\alpha-1}\left(-R^{-1}\ddot{\tau} + (\alpha+1)R^{-1}\tau^{-1}\dot{\tau}^2 + (n-1)r^{-1}\dot{\tau} + \sqrt{\kappa}(n-1)\dot{\tau}\right). \end{aligned} \tag{8.35}$$

Since the derivatives of $\tau(2 - r/R)$ vanish on $B(p,R)$, (8.35) vanishes on $B(p,R)$, so it can be supposed that $R \leq r < 2R$. Hence, since $\dot{\tau}$ is nonnegative,

$$\begin{aligned} &\phi^\sigma\left((\psi')^2 - \psi'' - (n-1)(r^{-1} + \sqrt{\kappa})\psi'\right) \\ &\quad \leq \alpha R^{-1}\tau^{\sigma\alpha-1}\left(-R^{-1}\ddot{\tau} + (\alpha+1)R^{-1}\tau^{-1}\dot{\tau}^2 + (n-1)R^{-1}\dot{\tau} + \sqrt{\kappa}(n-1)\dot{\tau}\right), \end{aligned} \tag{8.36}$$

on $B(p,2R)$. As long as $\sigma\alpha > 1$, the right-hand side of (8.36) is bounded on $B(p,2R)$. Choosing $\alpha = 2\sigma^{-1}$ yields that, on $B(p,2R)$,

$$\begin{aligned} &\phi^\sigma\left((\psi')^2 - \psi'' - (n-1)(r^{-1} + \sqrt{\kappa})\psi'\right) \\ &\quad \leq 2\sigma^{-1}R^{-1}\tau\left(-R^{-1}\ddot{\tau} + (2\sigma^{-1}+1)R^{-1}\tau^{-1}\dot{\tau}^2 + (n-1)R^{-1}\dot{\tau} + \sqrt{\kappa}(n-1)\dot{\tau}\right) \\ &\quad \leq C\sigma^{-1}R^{-1} \end{aligned} \tag{8.37}$$

for some $C > 0$ that depends only on n and $\sqrt{\kappa}$. Hence, since $\phi = 1$ on $B(p,R)$, substituting (8.37) in (8.29) yields

$$\begin{aligned} \sup_{B(p,R)} u^\sigma &= \sup_{B(p,R)} f^\sigma \leq \sup_{B(p,2R)} f^\sigma \\ &\leq \phi^\sigma\left(A + (\psi')^2 - \psi'' - (n-1)(r^{-1} + \sqrt{\kappa})\psi'\right) \leq B^{-1}(A + C\sigma^{-1}R^{-1}). \end{aligned} \tag{8.38}$$

There remains to consider the case $x_0 \in \{p\} \cup \mathsf{Cut}(p)$. First, suppose $x_0 = p$. That is, f attains its maximum over the closure of $B(p,2R)$ at p. Then

$$\sup_{B(p,R)} u = \sup_{B(p,R)} f \leq \sup_{B(p,2R)} f = f(p) = \phi(p)u(p) = u(p), \tag{8.39}$$

so u attains its maximum over the closer of $B(p,R)$ at p. Then at p there holds $0 \geq \Delta u = Bu^{1+\sigma} - Au$, so $A/B \geq u(p)^\sigma \geq \sup_{B(p,R)} u^\sigma$, and this yields the desired claim.

For the proof in the case $x_0 \in \mathsf{Cut}(p)$ the argument is modified as in the proof of Yau's gradient estimate on page 21 in Section I.3 of [37]. This is recalled here now. By the assumption that $x_0 \in \mathsf{Cut}(p)$ there is a minimizing geodesic joining p to x_0 the image σ of which necessarily lies in $B(p,R)$ (because no point of σ is farther from p than is x_0). Let $\bar{x}$ be a point on σ lying strictly between p and x_0 at some distance $\epsilon > 0$ from p. Since σ is minimizing, no point of the interior of σ can be conjugate to $\bar{x}$. Were p or x_0 conjugate to $\bar{x}$ then $\bar{x}$ would be in its cut locus, which it is not because $x_0 \in \mathsf{Cut}(p)$ (and so $p \in \mathsf{Cut}(x_0)$). Thus no point of σ is a conjugate point of $\bar{x}$ and hence there is some $\delta > 0$ for which there is an open δ neighborhood $N \subset B(p,R)$ of σ containing no conjugate point of $\bar{x}$. Let $\bar{r} = d(\bar{x}, \cdot\,)$. By the triangle inequality, $\bar{r} + \epsilon \geq r$. On the other hand $\bar{r}(x_0) + \epsilon = r(x_0)$. Define $\bar{f} = \phi(\bar{r} + \epsilon)u$. Then $\bar{f} \leq f$ on N and $\bar{f}(x_0) = f(x_0)$, so $\bar{f}$ attains its maximum value on N at x_0. As $\bar{r}$ is smooth near x_0 the preceding argument goes through with $\bar{f}$ in place of f, and letting $\epsilon \to 0$ at the end yields (8.2). □

Remark 8.3. Suppose $n = \dim M > 2$. If $u \in C^\infty(M)$ is positive, straightforward computations using (3.13) show that the scalar curvatures $R_{\tilde{g}}$ and R_g of the Riemannian metrics $\tilde{g} = u^{\frac{4}{n-2}}g$ and

g are related by

$$\Delta_g u = \tfrac{n-2}{4(n-1)}\left(-R_{\tilde{g}} u^{1+\frac{4}{n-2}} + R_g u\right). \tag{8.40}$$

When $n = \dim M = 2$, the scalar curvatures $R_{\tilde{g}}$ and R_g of the Riemannian metrics $\tilde{g} = e^u g$ and g are related by

$$\Delta_g u = -R_{\tilde{g}} e^u + R_g. \tag{8.41}$$

Corollary 8.4 (Corollary of Theorem 8.1). *Let M be an n-dimensional manifold. If g is a complete Riemannian metric M having nonnegative scalar curvature, then there is no metric conformal to g having scalar curvature bounded from above by a negative constant.*

Proof. First suppose $n > 2$. Had $\tilde{g} = u^{\frac{4}{n-2}} g$ scalar curvature $R_{\tilde{g}}$ satisfying $\sup_M R_{\tilde{g}} \leq -\kappa$ for some constant $\kappa > 0$, then, by (8.40) there would hold $\Delta_g u \geq \frac{\kappa(n-2)}{4(n-1)} u^{1+\frac{4}{n-2}}$, and, by Theorem 8.1 with $A = 0$, $\sup_M u = 0$. If $n = 2$, had $\tilde{g} = e^u g$ negative scalar curvature, then, since $e^x \geq \frac{1}{2}x^2$ for $x > 0$, by (8.41) there would hold $\Delta_g u \geq \frac{\kappa}{2} u^2$, and, by Theorem 8.1 with $A = 0$, $\sup_M u = 0$. □

Example 8.5. The following example illuminates the role of the completeness condition in Theorem 8.1. On the open ball $B(0, R)$ of radius $R > 0$ around the origin in Euclidean space $(\mathbb{R}^n, \delta_{ij})$, the function

$$u(x) = \left(R - R^{-1}|x|^2\right)^{\frac{2-n}{2}} \tag{8.42}$$

solves

$$\Delta u = n(n-2) u^{1+\frac{4}{n-2}}, \tag{8.43}$$

This does not contradict Theorem 8.1 because the restriction to $B(0, R)$ of the Euclidean metric is not complete, so it is not possible to take R to $+\infty$ in (8.2).

8.1. **Cheng-Yau local gradient estimate for harmonic functions.** A positive harmonic function on $\mathbb{R}^n$ must be constant. The proof requires something like the Harnack inequality, which will be deduced here as a consequence of the gradient estimate.

The gradient estimate for a positive harmonic function bounds the norm of the logarithmic derivative of such a harmonic function in terms of a lower bound on the Ricci curvature. This local estimate can be proved for a positive eigenfunction of the Laplacian with only a bit of extra work, so this more general version is given here. The basic result is due to S. T. Yau. The sharp form stated here is Lemma 2.1 in [28].

The proof given here is different than that usually given. It is based on Theorem 8.2.

Theorem 8.6 (Gradient estimate). *Let (M, g) be a complete Riemannian manifold. Suppose that on the ball $B(p, 2R)$ the Ricci curvature is bounded from below by $R_{ij} \geq -\kappa(n-1)g_{ij}$ for $\kappa \geq 0$. Suppose u is a smooth positive function harmonic on $B(p, 2R)$. There is a constant $C(n, \kappa) > 0$ depending only on n and κ such that*

$$\sup_{B(p,R)} |d \log u| \leq \sqrt{\kappa}(n-1) + CR^{-1} \tag{8.44}$$

for all $x \in B(p, R)$. If $R_{ij} \geq -\kappa(n-1)g_{ij}$ on all of M and u is defined on all of M, then

$$\sup_M |d \log u| \leq \sqrt{\kappa}(n-1). \tag{8.45}$$

Remark 8.7. In [31], O. Munteanu proves a refinement of the local gradient estimate (8.44), showing that there is a positive constant C_1 depending only on n and κ and a positive constant C_2 depending only on κ such that

$$\sup_{B(p,R)} |d\log u| \leq \sqrt{\kappa}(n-1) + \tfrac{C_1}{R}e^{-C_2 R}. \tag{8.46}$$

The proof follows the same lines as that of Theorem 8.6, but requires a better choice of test function and a more delicate analysis.

Proof of Theorem 8.6. Let $v = \log(u)$. The coefficient of $|dv|^{2\beta+2}$ in (4.62) is positive if $\beta > (n-2)/2$. In this case, applying Theorem 8.2 to (4.62) with $u = |dv|^{2\beta}$, $A = 2\beta\kappa(n-1)$, $B = \frac{2\beta(2\beta-(n-2)}{2(n-1)\beta-(n-2)}$, and $\sigma = \beta^{-1}$, shows that there is a constant $C_1 > 0$, depending only on n and κ, such that

$$\begin{aligned}\sup_{B(p,R)} |dv|^2 &= \sup_{B(p,R)} |dv|^{2\beta\sigma} \leq B^{-1}(A + C_1\sigma^{-1}R^{-1})\\ &= \tfrac{(2(n-1)\beta-(n-2))}{2\beta(2\beta-(n-2)}\left(2\beta\kappa(n-1) + C_1\beta R^{-1}\right)\\ &= \tfrac{(2(n-1)\beta-(n-2))}{(2\beta-(n-2)}\left(\kappa(n-1) + C_2 R^{-1}\right).\end{aligned} \tag{8.47}$$

Taking the limit of the right-hand side as $\beta \to \infty$ yields

$$|dv|^2 \leq \kappa(n-1)^2 + C_3 R^{-1}, \tag{8.48}$$

where $C_3 > 0$ is a constant depending only on n and κ.

By (8.48), at a point of $B(p,R)$ where $|dv| > \sqrt{\kappa}(n-1)$ there holds

(8.49)
$$\sqrt{\kappa}(n-1)(|dv| - \sqrt{\kappa}(n-1)) \leq (|dv| - \sqrt{\kappa}(n-1))(|dv| + \sqrt{\kappa}(n-1)) \ \leq \kappa(n-1)^2 + C_3R^{-1}.$$

and this shows

$$|dv| \leq \sqrt{\kappa}(n-1) + C_4 R^{-1} \tag{8.50}$$

on $B(p,R)$. □

Corollary 8.8 (S. T. Yau). *On a complete Riemannian manifold with nonnegative Ricci curvature, a positive harmonic function is constant.*

8.2. **Standard proof of the local gradient estimate.** The following proof of the local gradient estimate (with sharp constant terms) follows that in [28]. This might be characterized as the standard proof.

Second proof of Theorem 8.6. Define $v = \log u$ and $G = \phi|dv|^2$, where $\phi \in C^2(B(x,2R))$ is a positive function that will be specified later. Straightforward computations show

$$\begin{aligned} d|dv|^2 &= \phi^{-1}dG - \phi^{-2}Gd\phi,\\ \Delta G &= \phi\Delta|dv|^2 + \phi^{-1}G\Delta\phi + 2\phi^{-1}\langle d\phi, dG\rangle - 2\phi^{-2}G|d\phi|^2.\end{aligned} \tag{8.51}$$

By (4.52), with $a = 0$ and $\lambda = 0$,

$$\begin{aligned}\tfrac{1}{2}\Delta G &\geq \phi\left(\tfrac{1}{n-1}|dv|^4 - \tfrac{(n-2)}{n-1}v^p d_p|dv|^2 + \tfrac{n}{4(n-1)}|dv|^{-2}|d|dv|^2|^2 - \kappa(n-1)|dv|^2\right)\\ &\quad + \tfrac{1}{2}\phi^{-1}G\Delta\phi + \phi^{-1}\langle d\phi, dG\rangle - \phi^{-2}G|d\phi|^2\\ &= \tfrac{1}{n-1}\phi^{-1}G^2 - \tfrac{(n-2)}{n-1}\langle dv, dG\rangle + \tfrac{n-2}{n-1}\phi^{-1}G\langle dv, d\phi\rangle\\ &\quad + \tfrac{n}{4(n-1)}\phi G^{-1}\left|\phi^{-1}dG - \phi^{-2}Gd\phi\right|^2 - \kappa(n-1)G\\ &\quad + \tfrac{1}{2}\phi^{-1}G\Delta\phi + \phi^{-1}\langle d\phi, dG\rangle - \phi^{-2}G|d\phi|^2.\end{aligned} \tag{8.52}$$

Simplifying yields

$$\begin{aligned}\tfrac{1}{2}\Delta G \geq{}& \tfrac{1}{n-1}\phi^{-1}G^2 + \left(-\kappa(n-1) + \tfrac{1}{2}\phi^{-1}\Delta\phi - \tfrac{3n-4}{4(n-1)}\phi^{-2}|d\phi|^2\right)G \\ &- \tfrac{n-2}{n-1}\phi^{-2}\langle dv, dG\rangle + \tfrac{n-2}{n-1}\phi^{-1}G\langle dv, d\phi\rangle + \tfrac{n-2}{n-1}\phi^{-1}\langle d\phi, dG\rangle + \tfrac{n}{4(n-1)}G^{-1}|dG|^2.\end{aligned} \tag{8.53}$$

By the Cauchy-Schwarz inequality, for $\epsilon > 0$,

$$\phi^{-1}G\langle dv, d\phi\rangle \geq -\tfrac{1}{2}\phi^{-1}G(\epsilon\phi|dv|^2 - \epsilon^{-1}\phi^{-1}|d\phi|^2) = -\tfrac{1}{2}(\phi^{-1}G^2 + G\phi^{-2}|d\phi|^2). \tag{8.54}$$

In (8.53) this yields

$$\begin{aligned}\tfrac{1}{2}\Delta G \geq{}& \left(\tfrac{1}{n-1} - \tfrac{(n-2)\epsilon}{2(n-1)}\right)\phi^{-1}G^2 + \left(-\kappa(n-1) + \tfrac{1}{2}\phi^{-1}\Delta\phi - \left(\tfrac{3n-4}{4(n-1)} + \tfrac{n-2}{2(n-1)\epsilon}\right)\phi^{-2}|d\phi|^2\right)G \\ &- \tfrac{n-2}{n-1}\phi^{-2}\langle dv, dG\rangle + \tfrac{n-2}{n-1}\phi^{-1}\langle d\phi, dG\rangle + \tfrac{n}{4(n-1)}G^{-1}|dG|^2.\end{aligned} \tag{8.55}$$

Define the function ϕ as follows. The function

$$\tau(t) = \begin{cases} 0 & \text{if } t = 0, \\ \left(1 + \exp(\frac{2}{t} + \frac{2}{t-1})\right)^{-1} & \text{if } 0 < t < 1, \\ 1 & \text{if } t \geq 1, \end{cases} \tag{8.56}$$

is infinitely differentiable on $[0, \infty)$ with bounded first and second derivatives. Observe that $\tau(t) + \tau(1-t) = 1$. The derivative of τ is

$$\tau'(t) = \begin{cases} 0 & \text{if } t = 0, \\ \frac{\left(\frac{2}{t^2} + \frac{2}{(t-1)^2}\right)\exp(\frac{2}{t} + \frac{2}{t-1})}{\left(1 + \exp(\frac{2}{t} + \frac{2}{t-1})\right)^2} & \text{if } 0 < t < 1, \\ 0 & \text{if } t \geq 1, \end{cases} \tag{8.57}$$

As t approaches 0 from above or 1 from below, the derivative $\tau'(t)$ tends to 0, from which it follows that $\tau'(t)$ is bounded on $[0, \infty)$. That τ'' is bounded can be checked similarly. Note also that τ' is positive on $(0, 1)$. Define $\phi(t) = \tau(2 - t/R)^2$, so that

$$\phi(t) = \begin{cases} 1 & \text{if } t \leq R, \\ \left(1 + \exp\left(\frac{2}{2 - \frac{t}{R}} + \frac{2}{1 - \frac{t}{R}}\right)\right)^{-2} & \text{if } R < t < 2R, \\ 0 & \text{if } t \geq 2R, \end{cases} \tag{8.58}$$

Then $\phi' = -2R^{-1}\tau\tau' = -2R^{-1}\phi^{1/2}\tau'$, and since τ' is bounded, there is a constant $C > 0$ such that

$$0 \geq \phi^{-1/2}\phi' \geq -CR^{-1} \tag{8.59}$$

on $[R, 2R]$. Note that ϕ' is negative on $(R, 2R)$, and nonpositive on $[0, \infty)$. Similarly, $\phi'' = 2R^{-2}((\tau')^2 + \tau\tau'')$, and, because $(\tau')^2 + \tau\tau''$ is bounded, there is a constant $C > 0$ such that

$$|\phi''| \leq CR^{-2}, \tag{8.60}$$

on $[R, 2R]$. By the Laplacian comparison theorem, (8.59), and (8.60), at the points of $B(p, 2R)$ where r is smooth there holds

$$\begin{aligned}\Delta\phi = \phi'\Delta r + \phi'' &\geq (n-1)(r^{-1} + \sqrt{\kappa})\phi' + \phi'' \\ &\geq -C_1(n-1)(R^{-1} + \sqrt{\kappa})R^{-1} - C_2R^{-2} \\ &\geq -C_3(R^{-2} + \sqrt{\kappa}R^{-1}),\end{aligned} \tag{8.61}$$

where here and in what follows C_i indicates a positive constant depending only on n, and in the second inequality of (8.61) there is used that $\phi \leq 1$. By (8.59),

$$\phi^{-2}|d\phi|^2 \leq C_4 R^{-2}. \tag{8.62}$$

By the construction of ϕ, the function G attains a maximum at some point $x_0 \in B(p, 2R)$. Suppose $x_0 \notin \{\{p\} \cup \mathsf{Cut}(p)\}$, so that r is smooth at x_0. The case where x_0 is a cut point of p can be treated by using a barrier function of the form $d(\gamma(\epsilon), \cdot\,) + \epsilon$, where γ is a minimizing geodesic from p to x_0, as in the proof of Theorem 8.1; the details will be omitted. In the case where $x_0 = p$, then $G(p) = |dv|(p)^2$, and in this case a straightforward argument using the nonpositivity at p of $\Delta|dv|^2$ yields the desired estimate.

By (8.55), at x_0 there holds

$$\begin{aligned} 0 &\geq \tfrac{1}{2}\Delta G \\ &\geq \left(\tfrac{1}{n-1} - \tfrac{(n-2)\epsilon}{2(n-1)}\right)\phi^{-1}G^2 + \left(-\kappa(n-1) + \tfrac{1}{2}\phi^{-1}\Delta\phi - \left(\tfrac{3n-4}{4(n-1)} + \tfrac{n-2}{2(n-1)\epsilon}\right)\phi^{-2}|d\phi|^2\right)G. \end{aligned} \tag{8.63}$$

Substituting (8.61) and (8.62) in (8.63), using $\phi \leq 1$, supposing $\epsilon(n-2) < 2$, and rearranging and simplifying terms yields

$$\begin{aligned} \tfrac{2-(n-2)\epsilon}{2(n-1)}G &\leq \tfrac{2-(n-2)\epsilon}{2(n-1)}\phi^{-1}G \\ &\leq \kappa(n-1) + \tfrac{1}{2}C_4(R^{-2} + \sqrt{\kappa}R^{-1}) + \tfrac{(3n-4)\epsilon+2(n-2)}{4(n-1)}C_3R^{-2} \\ &\leq \kappa(n-1) + \tfrac{\sqrt{\kappa}}{2}R^{-1} + (C_6 + C_7\epsilon^{-1})R^{-2}. \end{aligned} \tag{8.64}$$

Since the right-hand side of (8.64) does not depend on x_0, and G attains its maximum on $B(p, 2R)$ at x_0, the inequality (8.64) is valid on all of $B(p, 2R)$. Hence

$$\tfrac{2-(n-2)\epsilon}{2}G \leq \kappa(n-1)^2 + \tfrac{\sqrt{\kappa}(n-1)}{2}R^{-1} + (n-1)(C_6 + C_7\epsilon^{-1})R^{-2}, \tag{8.65}$$

for all $x \in B(p, 2R)$. Since ϕ equals 1 on $B(p, R)$, there results

$$(2-(n-2)\epsilon)|dv|^2 \leq 2\kappa(n-1)^2 + \sqrt{\kappa}(n-1)R^{-1} + 2(n-1)(C_6 + C_7\epsilon^{-1})R^{-2}, \tag{8.66}$$

for $x \in B(p, R)$. Choose $\epsilon = (R + n - 2)^{-1}$. Then, for $x \in B(p, R)$,

$$\tfrac{2R+n-2}{R+n-2}|dv|^2 \leq 2\kappa(n-1)^2 + C_8R^{-1} + C_9R^{-2}, \tag{8.67}$$

where C_8 and C_9 depend only on n and κ. Hence, for $x \in B(p, R)$,

$$|dv|^2 \leq \kappa(n-1)^2\left(1 + \tfrac{n-2}{2R+n-2}\right) + C_{10}R^{-1} + C_{11}R^{-2} \leq \kappa(n-1)^2 + C_{12}R^{-1}. \tag{8.68}$$

By (8.68), at a point of $B(p, R)$ where $|dv| > \sqrt{\kappa}(n-1)$ there holds

$$\sqrt{\kappa}(n-1)(|dv| - \sqrt{\kappa}(n-1)) \leq (|dv| - \sqrt{\kappa}(n-1))(|dv| + \sqrt{\kappa}(n-1)) \leq \kappa(n-1)^2 + C_{12}R^{-1}. \tag{8.69}$$

and this shows

$$|dv| \leq \sqrt{\kappa}(n-1) + C_{13}R^{-1} \tag{8.70}$$

on $B(p, R)$. □

Example 8.9. A linear function on $\mathbb{R}^n$ is harmonic. On the domain where it is positive, the norm of its logarithmic derivative is the reciprocal of a linear function.

Example 8.10. On $\mathbb{R}^n \setminus \{0\}$ the nonconstant function $u(x) = |x|^{2-n}$ is positive and harmonic (the function $G(x, y) = u(x - y)$ is the Green's function of the Euclidean Laplacian).

Corollary 8.11 (Harnack inequality for harmonic functions). *Let (M, g) be a complete n-dimensional Riemannian manifold with Ricci curvature satsifying $R_{ij} \geq -\kappa(n-1)g_{ij}$. There is a constant $C = C(n, \kappa, R)$, depending only on n, κ, and $R > 0$, such that if u is a positive harmonic function on the geodesic ball $B(p, R)$ then*

$$\sup_{B(p,R/2)} u \leq C \inf_{B(p,R/2)} u. \tag{8.71}$$

Proof. Let x and y be any two distinct points in $B(p, R/2)$ and let γ be a minimal geodesic joining x to y. By the triangle inequality, for any point z on γ there holds

$$d(z,p) \leq \tfrac{1}{2}(d(z,x) + d(x,p)) + \tfrac{1}{2}(d(z,y) + d(y,p)) = \tfrac{1}{2}(d(x,y) + d(x,p) + d(y,p)) \leq R, \tag{8.72}$$

so γ is contained in $B(p, R)$. By the gradient estimate in $B(p, 2R)$, there is a positive constant c depending only on n, κ, and R, such that

$$\begin{aligned} |\log u(y) - \log u(x)| &= \left| \int_\gamma d\log u(\dot{\gamma}(s))\, ds \right| \leq \int_\gamma |d\log u|\, ds \\ &\leq \int_\gamma ((n-1)\sqrt{\kappa} + cR^{-1})\, ds \leq 2(n-1)\sqrt{\kappa}R + 2c = \log C(n, \kappa, R). \end{aligned} \tag{8.73}$$

Hence $u(y) \leq Cu(x)$ for all $x, y \in B(p, R/2)$. Since this is true for all y for any fixed x, it implies $\sup_{B(p,R/2)} u \leq Cu(x)$ for all $x \in B(p, R/2)$. The claim follows taking the infimum of the left-hand side of this last inequality. □

8.3. **Harmonic functions of polynomial growth on Euclidean space.** A function f on a complete Riemannian manifold (M, g) has **polynomial growth of order** k if there is $p \in M$ and a constant $C > 0$ such that

$$f(q) \leq Cd(p,q)^k \tag{8.74}$$

for all $q \in M$. Since, by the triangle inequality $d(\bar{p}, q)^k$ is bounded by $d(p,q)^k$ multiplied by a degree k polynomial in $d(p, \bar{p})$, this condition is true for all $p \in M$ (with possibly different constants C) if it is true for some $p \in M$.

Since harmonic functions on a compact manifold are constant, the space of harmonic functions of order k polynomial growth has dimension 1 for all $k \geq 1$. A noncompact complete Riemannian manifold for which the dimensions of the spaces of harmonic functions of order k polynomial growth are finite can therefore be seen as behaving in some respects like a compact manifold. The gradient estimate is a useful tool for obtaining bounds on these dimensions.

While there is much to say with regards to these questions, here there is shown only the simplest case, namely that of Euclidean space.

Theorem 8.12. *Let (M, g) be a complete n-dimensional Riemannian manifold with nonnegative Ricci curvature. Any harmonic function u satisfying*

$$\liminf_{x \to \infty} d(p,x)^{-1}u(x) \geq 0 \tag{8.75}$$

for some $p \in M$ is constant.

Proof. Let $i(R) = \inf_{B(p,R)} u$. By the maximum principle, u attains its infimum over $B(p, R)$ at some point x_0 of the boundary $\partial B(p, R)$, so $u - i(R)$ is a positive harmonic function on $B(p, R)$. Moreover, $R^{-1}i(R) = r(x_0)^{-1}u(x_0)$, so the condition (8.75) implies that $\lim_{R\to\infty} R^{-1}i(R) \geq 0$. By the local gradient estimate, Theorem 8.6, applied to $u - i(2R)$ on $B(p, 2R)$,

$$|d(u - i(2R)|(x) \leq CR^{-1}(u(x) - i(2R)) \tag{8.76}$$

for all $x \in B(p, R)$, for a constant $C > 0$ that depends only on n. Letting $R \to \infty$ in (8.76) shows that u is constant. □

Remark 8.13. If the condition (8.75) is true for some $p \in M$, it is true for all $p \in M$.

Remark 8.14. Theorem 8.12 implies that a nonnegative harmonic function of linear growth on Euclidean space is a constant. It also implies that any harmonic function of sublinear growth on Euclidean space is constant.

Theorem 8.15. *A harmonic function of polynomial growth of order k on Euclidean space is a harmonic polynomial of degree at most k.*

Proof. Because Euclidean space is flat, the partial derivatives of any order of a harmonic function are harmonic functions. It is straightforward to check that a partial derivative of order l of a function of polynomial growth of order k is a function of polynomial growth of order $k - l$. In particular, the $k$fold partial derivatives of a harmonic function of polynomial growth of order k are harmonic functions of polynomial growth of order 0. By Theorem 8.12, these partial derivatives must be constant. It follows that the harmonic function is annihilated by all partial derivatives of order $k + 1$, so must be a polynomial of degree at most k. □

8.4. **Nonexistence of L^p harmonic functions on a complete Riemannian manifold.** To motivate the results described here, recall the following result abou the growth of harmonic functions on Euclidean space.

Theorem 8.16. *Let u be a positive subharmonic function on $\mathbb{R}^n$ such that $u \in L^p(\mathbb{R}^n)$ for some $1 \leq p < \infty$. Then u is identically zero.*

Proof. By the mean value inequality for subharmonic functions,

$$u(x) \leq \frac{\int_{B(x,R)} u(y)\, dy}{\mathsf{vol}(B(x, R))}. \tag{8.77}$$

By the Hölder inequality,

$$u(x)^p \leq \left(\frac{\int_{B(x,R)} u(y)\, dy}{\mathsf{vol}(B(x, R))}\right)^p \leq \frac{\int_{B(x,R)} u(y)^p\, dy}{\mathsf{vol}(B(x, R))^{p-p/q}} = \frac{\int_{B(x,R)} u(y)^p\, dy}{\mathsf{vol}(B(x, R))} \tag{8.78}$$

where $p^{-1} + q^{-1} = 1$. If $u \in L^p(\mathbb{R}^n)$, letting $R \to \infty$ yields $u(x) = 0$. □

Corollary 8.17. *Let u be a harmonic function on $\mathbb{R}^n$ such that $u \in L^p(\mathbb{R}^n)$ for some $1 \leq p < \infty$. Then u is identically zero.*

Proof. Because u is harmonic, $|u|$ is subharmonic. □

Yau extended this result to an arbitrary complete Riemannian manifold.

Theorem 8.18 (S. T. Yau)**.** *On a complete Riemannian manifold a harmonic function or a positive subharmonic function u contained in L^p for some $1 < p < \infty$ is constant. In particular, if the manifold has infinite volume, u is identically zero.*

8.5. **Estimates on first eigenfunction of the Laplacian.** Theorem 8.20 gives a sharp bound on a positive eigenfunction of the Laplacian on a complete Riemannian manifold with a lower Ricci curvature bound. This theorem is given as Theorem 6.1 in [27]. It was also published in [6]. It is not clear who first obtained the sharp estimate. The proof given here is different than the proofs in the cited references. The proof uses the Omori-Yau maximum principle, which is recalled now.

Theorem 8.19 (Omori-Yau maximum principle)**.** *Let (M, g) be an n-dimensional complete Riemannian manifold with Ricci curvature satisfying $R_{ij} \geq -\kappa(n - 1)g_{ij}$ for some constant $\kappa \geq 0$. If $f \in C^2(M)$ is bounded from above then there is a sequence of points $\{x_k\} \subset M$ such that*

$$f(x_k) > \sup_M f - \tfrac{1}{k}, \qquad |df(x_k)| < \tfrac{1}{k}, \qquad \Delta f(x_k) < \tfrac{1}{k}. \tag{8.79}$$

For a proof of Theorem 8.19 see [34], where the result is obtained under weaker hypotheses.

Theorem 8.20. *Let (M,g) be an n-dimensional complete Riemannian manifold with Ricci curvature satisfying $R_{ij} \geq -\kappa(n-1)g_{ij}$ for some constant $\kappa \geq 0$. If a positive function $u \in C^\infty(M)$ solves $\Delta u = \lambda u$ for some constant $\lambda \leq 0$, then*

$$\sup_M |d\log u|^2 \leq \tfrac{\kappa(n-1)^2}{2} + \lambda + \sqrt{\kappa}(n-1)\sqrt{\tfrac{\kappa(n-1)}{4} + \lambda}, \tag{8.80}$$

and

$$-\lambda \leq \tfrac{\kappa(n-1)^2}{4}. \tag{8.81}$$

Proof. Let $v = \log u$ and $w = |dv|^2 - \lambda = -\Delta v$ Taking $f = v$ in (4.46) yields

$$\begin{aligned}\tfrac{1}{2}\Delta|dv|^2 &\geq \tfrac{1}{n-1}w^2 + \tfrac{2}{n-1}w|dv|^{-1}v^p d_p|dv| - v^p w_p + \tfrac{n}{n-1}|d|dv||^2 - \kappa(n-1)|dv|^2\\ &\geq \tfrac{1}{n-1}w^2 + \tfrac{2}{n-1}w|dv|^{-1}v^p d_p|dv| - 2|dv|vpd_p|dv| + \tfrac{n}{n-1}|d|dv||^2 - \kappa(n-1)|dv|^2\\ &\geq \tfrac{1}{n-1}w^2 - \tfrac{2(n-2)}{n-1}|dv|v^p d_p|dv| - \tfrac{2}{n-1}\lambda|dv|vpd_p|dv| + \tfrac{n}{n-1}|d|dv||^2 - \kappa(n-1)|dv|^2\\ &\geq \tfrac{1}{n-1}w^2 - \tfrac{2(n-2)}{n-1}|dv|^2|d|dv|| + \tfrac{2}{n-1}\lambda|d|dv|| + \tfrac{n}{n-1}|d|dv||^2 - \kappa(n-1)|dv|^2,\end{aligned} \tag{8.82}$$

where the last step follows from the Cauchy-Schwarz inequality and nonpositivity of λ. For β sufficiently large,

(8.83)

$$\begin{aligned}&\tfrac{1}{2\beta}|dv|^{2(1-\beta)}\Delta|dv|^{2\beta} = \tfrac{1}{2}\Delta|dv|^2 + \tfrac{1}{2}(\beta-1)|dv|^{-2}|d|dv|^2\\ &= \tfrac{1}{2}\Delta|dv|^2 + 2(\beta-1)|d|dv||^2\\ &\geq \tfrac{1}{n-1}|dv|^4 - \tfrac{2\lambda}{n-1}|dv|^2 + \tfrac{\lambda^2}{n-1} - \tfrac{2(n-2)}{n-1}|dv|^2|d|dv|| + \tfrac{2}{n-1}\lambda|d|dv||\\ &\qquad + \tfrac{2(n-1)\beta-(n-2)}{n-1}|d|dv||^2 - \kappa(n-1)|dv|^2\\ &\geq \tfrac{1}{n-1}\left(1 - \tfrac{n-2}{n-1}\epsilon\right)|dv|^4 + \left(\tfrac{2(n-1)\beta-(n-2)}{n-1} - \tfrac{n-2}{n-1}\tfrac{1}{\epsilon}\right)|d|dv||^2 + \tfrac{1}{n-1}\left(|d|dv| + \lambda\right)^2\\ &\qquad - \tfrac{1}{n-1}|d|dv||^2 - \kappa(n-1)|dv|^2\\ &\geq \tfrac{1}{n-1}\left(1 - \tfrac{n-2}{n-1}\epsilon\right)|dv|^4 + \left((2\beta-1) - \tfrac{n-2}{n-1}\tfrac{1}{\epsilon}\right)|d|dv||^2 + \tfrac{1}{n-1}\left(|d|dv| + \lambda\right)^2 - \kappa(n-1)|dv|^2,\end{aligned}$$

where the penultimate inequality uses the Cauchy-Schwarz inequality in the form $2ab \leq \epsilon a^2 + \epsilon^{-1}b^2$. Choosing $\epsilon = \frac{n-2}{(n-1)(2\beta-1)}$ yields

$$\begin{aligned}&\tfrac{1}{2\beta}|dv|^{2(1-\beta)}\Delta|dv|^{2\beta}\\ &\geq \tfrac{1}{n-1}\left(1 - \tfrac{(n-2)^2}{(n-1)^2(2\beta-1)}\right)|dv|^4 + \tfrac{1}{n-1}\left(|d|dv| + \lambda\right)^2 - \kappa(n-1)|dv|^2\\ &\geq \tfrac{1}{n-1}\left(1 - \tfrac{(n-2)^2}{(n-1)^2(2\beta-1)}\right)|dv|^4 - \kappa(n-1)|dv|^2.\end{aligned} \tag{8.84}$$

Hence

$$\Delta|dv|^{2\beta} \geq \tfrac{2\beta}{n-1}\left(1 - \tfrac{(n-2)^2}{(n-1)^2(2\beta-1)}\right)|dv|^{2\beta+2} - 2\beta\kappa(n-1)|dv|^{2\beta}. \tag{8.85}$$

By Theorem 8.1 with $\sigma = \beta^{-1}$,

$$\sup_M |dv|^2 = (\sup_M |dv|^{2\beta})^\sigma \leq \tfrac{n-1}{2\beta}\left(1 - \tfrac{(n-2)^2}{(n-1)^2(2\beta-1)}\right)^{-1} 2\beta\kappa(n-1) \tag{8.86}$$

Letting $\beta \to \infty$ yields

$$\sup_M |dv|^2 = (\sup_M |dv|^{2\beta})^\sigma \leq \kappa(n-1)^2. \tag{8.87}$$

This is not the desired bound, but suffices to establish that $|dv|^2$, and so also w, is bounded. Hence the Omori-Yau maximum principle can be applied to w. First it is necessary to rewrite (8.82) in terms of w.

By (8.82),

$$\begin{aligned}\tfrac{1}{2}\Delta w &\geq \tfrac{1}{n-1}w^2 - \tfrac{2(n-2)}{n-1}(w+\lambda)|d|dv|| + \tfrac{2\lambda}{n-1}|d|dv|| + \tfrac{n}{n-1}|d|dv||^2 - \kappa(n-1)(w+\lambda)\\ &\geq \tfrac{1}{n-1}w^2 - \tfrac{2(n-2)}{n-1}w|d|dv|| - \tfrac{2(n-3)\lambda}{n-1}|d|dv|| + \tfrac{n}{n-1}|d|dv||^2 - \kappa(n-1)(w+\lambda).\end{aligned} \tag{8.88}$$

Hence, since $|dw|^2 = 4(w+\lambda)|d|dv||^2$,

$$\begin{aligned}\tfrac{1}{2\beta}w^{1-\beta}\Delta w^{\beta} &= \tfrac{1}{2}\Delta w + 2(\beta-1)w^{-1}(w+\lambda)|d|dv||^2\\ &\geq \tfrac{1}{n-1}w^2 - \tfrac{2(n-2)}{n-1}w|d|dv|| + \tfrac{2(n-1)\beta-(n-2)}{n-1}|d|dv||^2 - \tfrac{2(n-3)\lambda}{n-1}|d|dv||\\ &\quad + 2(\beta-1)\lambda w^{-1}|d|dv||^2 - \kappa(n-1)(w+\lambda)\\ &\geq \tfrac{1}{n-1}\left(1 - \tfrac{n-2}{n-1}\epsilon\right)w^2 + \left(\tfrac{2(n-1)\beta-(n-2)}{n-1} - \tfrac{n-2}{n-1}\tfrac{1}{\epsilon}\right)|d|dv||^2 - \tfrac{2(n-3)\lambda}{n-1}|d|dv||\\ &\quad + 2(\beta-1)\lambda w^{-1}|d|dv||^2 - \kappa(n-1)(w+\lambda)\end{aligned} \tag{8.89}$$

Choosing $\epsilon = \frac{n-2}{2(n-1)\beta-(n-2)}$ yields

$$\tfrac{1}{2\beta}w^{1-\beta}\Delta w^{\beta} \geq \tfrac{1}{n-1}\left(1 - \tfrac{(n-2)^2}{2(n-1)\beta-(n-2)}\right)w^2 - \tfrac{2(n-3)\lambda}{n-1}|d|dv|| + 2(\beta-1)\lambda w^{-1}|d|dv||^2 - \kappa(n-1)(w+\lambda) \tag{8.90}$$

By the Omori-Yau maximum principle, given $\epsilon > 0$ there is $x \in M$ such that

$$|dv(x)|^2 > \sup_M |dv|^2 - \epsilon, \quad 4|dv|(x)^2|d|dv|(x)| = |d|dv|^2|(x) < \epsilon, \quad \Delta w(x) = \Delta|dv|^2(x) < \epsilon. \tag{8.91}$$

Let $W = \sup_M w$. Then $w(x) > W - \epsilon$ and

$$\begin{aligned}|d|dv||(x) &< \tfrac{\epsilon}{4|dv|(x)^2} < \tfrac{\epsilon}{4(\sup_M |dv|^2-\epsilon)},\\ \tfrac{1}{2\beta}w(x)^{1-\beta}(\Delta w(x))^{\beta} &= \tfrac{1}{2}\Delta w(x) + \tfrac{1}{2}(\beta-1)|dw(x)|^2 \leq \tfrac{\epsilon\beta}{2}.\end{aligned} \tag{8.92}$$

Evaluating (8.90) at x using (8.92) yields

$$\tfrac{\epsilon\beta}{2} \geq \tfrac{1}{n-1}\left(1 - \tfrac{(n-2)^2}{2(n-1)\beta-(n-2)}\right)(W-\epsilon)^2 + \tfrac{2(\beta-1)\lambda\epsilon^2}{16(W-\epsilon)(\sup_M |dv|^2-\epsilon)^2} - \kappa(n-1)(W+\lambda). \tag{8.93}$$

Letting $\epsilon \to 0$ in (8.93) yields

$$0 \geq \tfrac{1}{n-1}\left(1 - \tfrac{(n-2)^2}{2(n-1)\beta-(n-2)}\right)W^2 - \kappa(n-1)(W+\lambda). \tag{8.94}$$

Letting $\beta \to \infty$ in (8.94) yields

$$0 \geq W^2 - \kappa(n-1)^2 W - \kappa(n-1)^2\lambda. \tag{8.95}$$

For the right-hand side of (8.95) to be nonpositive it must be that $-\lambda \leq \frac{\kappa(n-1)^2}{4}$. The claim follows. $\square$

9. First eigenvalue of the Laplacian

Basic problems include understanding the first eigenvalue of the Laplacian on a compact manifold, the bottom of the spectrum of the L^2 Laplacian on a complete noncompact manifold, and the structure and growth properties of harmonic functions and eigenfunctions of the Laplacian.

The maximum principle and the gradient estimate are fundamental tools in obtaining such information.

9.1. **First eigenvalue of the Laplacian.** Let (M, g) be a compact Riemannian operator. Then the Laplacian Δ is a compact self-adjoint operator (reference?), so by the spectral theorem for compact self-adjoint operators, has discrete spectrum. A nontrivial eigenvalue of Δ is necessarily negative, for if $\Delta u = \lambda u$, then integration by parts shows

$$\lambda \int_M u^2 \, d\mathrm{vol}_g = \int_M u \Delta u \, d\mathrm{vol}_g = -\int_M |du|^2 \, d\mathrm{vol}_g < 0. \tag{9.1}$$

The smallest positive real number μ such that $\Delta u = -\mu u$ has a solution is called the *first eigenvalue* of the Laplacian (although it is the negative of an eigenvalue of the Laplacian) and is written λ_1. More generally, one defines in the same way the mth eigenvalue of the Laplacian and one writes λ_m.

If $\Delta u = \lambda u$ then integration parts shows

$$\lambda \int_M u \, d\mathrm{vol}_g = \int_M \Delta u \, d\mathrm{vol}_g = 0, \tag{9.2}$$

so the eigenfunction u necessarily changes sign on M.

9.2. **Lower bound on first eigenvalue given a positive lower bound on Ricci curvature.** The lower bound in Theorem 9.1 is due to A. Lichnerowicz.

Theorem 9.1. *Let (M, g) be a compact n-dimensional Riemannian manifold with diameter d and having Ricci curvature satisfying $R_{ij} \geq \kappa(n-1)g_{ij}$ for some $\kappa > 0$. Then*

$$\lambda_1 \geq \kappa n, \tag{9.3}$$

with equality if and only if (M, g) is a round sphere.

Proof. Let u satisfy $\Delta u = -\lambda_1 u$, write $\lambda = \lambda_1$, and define $g = |du|^2 + cu^2$ where the positive constant c will be specified later. There holds

$$\begin{aligned}
\tfrac{1}{2}\Delta g &= \tfrac{1}{2}\Delta |du|^2 + cu\Delta u + c|du|^2 \\
&= |Ddu|^2 + u^p(\Delta u)_p + u^p u^q R_{pq} - c\lambda u^2 + c|du|^2 \\
&= |Ddu|^2 + (c - \lambda)|du|^2 + u^p u^q R_{pq} - c\lambda u^2 \\
&= |Ddu|^2 - \tfrac{\lambda^2}{n}u^2 + \lambda(\tfrac{\lambda}{n} - c)u^2 + (c - \lambda)|du|^2 + u^p u^q R_{pq} \\
&\geq |Ddu|^2 - \tfrac{\lambda^2}{n}u^2 + \lambda(\tfrac{\lambda}{n} - c)u^2 + (c - \lambda + \kappa(n-1))|du|^2.
\end{aligned} \tag{9.4}$$

For any $f \in C^\infty(M)$ there holds

$$0 \leq |Ddf - \tfrac{1}{n}(\Delta f)g|^2 = |Ddf|^2 - \tfrac{1}{n}(\Delta f)^2. \tag{9.5}$$

Choosing $c = \lambda/n$ in (9.4) yields

$$\tfrac{1}{2}\Delta g \geq |Ddu|^2 - \tfrac{\lambda^2}{n}u^2 + (n-1)(\kappa - \tfrac{\lambda}{n})|du|^2 \geq (n-1)(\kappa - \tfrac{\lambda}{n})|du|^2, \tag{9.6}$$

the last inequality by (9.5). If $\lambda \leq \kappa n$, then g is subharmonic, and since M is compact, g must be constant. If $\lambda < \kappa n$, this yields a contradiction in (9.6), because u is not constant. Hence $\lambda \geq \kappa n$. Moreover, if $\lambda = \kappa n$, then g equals some constant τ. Since

$$|du|^2 + \tfrac{\lambda}{n}u^2 = \tau, \tag{9.7}$$

τ must be positive. At a critical point of u there holds $u^2 = \tau n/\lambda$ so $m = \max_M u = -\min_M u = \sqrt{\tau n/\lambda}$. Replacing u by u/m it can be assumed from the beginning that $\max_M u = 1 = -\min_M u$.

Then $|du|^2 = \frac{\lambda}{n}(1-u^2) = \kappa(1-u^2)$. Let γ be a minimizing geodesic that runs from a point where u assumes its minimum to a point where u assumes its maximum. Then

$$d\sqrt{\kappa} \geq \int_\gamma \tfrac{|du|}{\sqrt{1-u^2}} \geq \int_{-1}^{1} \tfrac{du}{\sqrt{1-u^2}} = \pi. \tag{9.8}$$

By a theorem of S. Y. Cheng, if a complete Riemannian n-manifold has Ricci curvature bounded from below by $(n-1)\kappa > 0$ and diameter $\pi/\sqrt{\kappa}$, then it is isometric to a round sphere. □

Example 9.2 (Harmonic polynomials and spectrum of Laplacian on $\mathbb{S}^{n-1}$)**.** The simplest compact manifold is the round sphere.

Let $\mathsf{Pol}^k(\mathbb{V})$ denote the vector space of homogeneous degree k polynomials on the n-dimensional real vector space $\mathbb{V}$. Polarization yields a linear isomorphism between $\mathsf{Pol}^k(\mathbb{V})$ and $S^k(\mathbb{V}^*)$; its inverse is given by $\omega_{i_1\dots i_k} \to P^\omega(x) = \omega_{i_1\dots i_k}\mathbb{X}^{i_1}\dots\mathbb{X}^{i_k}$ where $\mathbb{X}$ is the radial Euler vector field. For $k \geq 2$, the Euclidean Laplacian Δ is a linear operator $\mathsf{Pol}^k(\mathbb{R}^n)$ to $\mathsf{Pol}^{k-2}(\mathbb{R}^n)$. The corresponding linear operator $\mathsf{tr} : S^k(\mathbb{V}^*) \to S^{k-2}(\mathbb{V}^*)$ is given by contraction with the metric, $\omega_{i_1\dots i_k} \to \mathsf{tr}(\omega)_{i_1\dots i_{k-2}} = \omega_{i_1\dots i_{k-2}p}{}^p$. Because

$$\binom{k}{2}\delta^{i_{k-1}i_{k-2}}\alpha_{(i_1\dots i_{k-2}}\delta_{i_{k-1}i_k)} = (n+2(k-1))\alpha_{i_1\dots i_{k-2}} + \binom{k-2}{2}\mathsf{tr}(\alpha)_{i_1\dots i_{k-4}}h_{i_{k-3}i_{k-2})}, \tag{9.9}$$

if $\mathsf{tr}(\alpha) = 0$ then α is in the image of tr, showing that tr is surjective. Hence, the space $\mathsf{Har}^k(\mathbb{V}) = \ker \mathsf{tr}$, of degree k harmonic polynomials on $\mathbb{V}$, has dimension

$$\dim S^k(\mathbb{V}^*) - \dim S^{k-2}(\mathbb{V}^*) = \binom{n+k-1}{k} - \binom{n+k-3}{k-2}. \tag{9.10}$$

This shows that there is an abundance of harmonic polynomials on $\mathbb{V}$.

Consider an n-dimensional Euclidean space $(\mathbb{V}, \delta)$ with Levi-Civita connection ∇. Let $\mathbb{S}^{n-1} = \{v \in \mathbb{V} : |v| = 1\}$ with the induced metric g and Levi-Civita connection D. Let $\mathbb{X}$ be the radial Euler vector field on $\mathbb{V}$ generating dilations around the origin, so that $r = |\mathbb{X}|$ is the distance from the origin and $\mathbb{X} = r\partial_r$. Let X and Y be vector fields such that $dr(X) = 0 = dr(Y)$, so that X and Y are tangent to the level sets of r. Let f be a C^2 function on $\mathbb{S}^{n-1}$. Then

$$\begin{aligned}(D_X df)(Y) &= Xdf(Y) - df(D_XY) = -df(\nabla_XY - h(X,Y)N)\\ &= (\nabla_X df)(Y) + h(X,Y)df(N) = (\nabla_X df)(Y) - r^{-1}g(X,Y)f_r,\end{aligned} \tag{9.11}$$

where $N = -|\mathbb{X}|^{-1}\mathbb{X} = -\partial_r$ is the inward normal, and h is the corresponding representative of the second fundamental form of $\mathbb{S}^{n-1}$. From (9.11) it follows that, along $\{v \in \mathbb{V} : |v| = r\}$ there holds

$$r^{-2}\Delta_g f = -(n-1)r^{-1}f_r + (\Delta_\delta f - \operatorname{Hess} f(N,N)) = \Delta_\delta f - (n-1)r^{-1}f_r - f_{rr}, \tag{9.12}$$

where g is the induced metric on $\mathbb{S}^{n-1}$. The induced metric Rewriting this yields

$$\Delta_{\text{euc}} f = f_{rr} + (n-1)r^{-1}f_r + r^{-2}\Delta_{\mathbb{S}^{n-1}}f = r^{1-n}(r^{n-1}f)_r + r^{-2}\Delta_{\mathbb{S}^{n-1}}f. \tag{9.13}$$

If P is a homogeneous polynomial of degree k, then $rP_r = dP(\mathbb{X}) = kP$ and $r^2P_{rr} = (\nabla_{\mathbb{X}}dP)(\mathbb{X}) = k(k-1)P$, so, if $P \in \mathsf{Har}^k(\mathbb{V})$, then

$$\Delta_{\mathbb{S}^{n-1}}P = -k(n+k-2)P. \tag{9.14}$$

Note that

$$-k(n-2+k) = -(k + \tfrac{n-2}{2})^2 + (\tfrac{n-2}{2})^2 < 0. \tag{9.15}$$

A bit more work is required to show that every eigenfunction of the spherical Laplacian arises in this way. Once this is established, it follows that the first eigenvalue of the n-dimensional sphere of radius R is unit sphere is nR^{-2}.

Using Corollary 4.21 and the maximum principle, there can be deduced the following estimate on the first eigenvalue of the Laplacian on a compact Riemannian manifold having Ricci curvature bounded from below by a negative constant. The proof follows closely the original one of Li and Yau in [29].

Theorem 9.3 (P. Li and S. T. Yau; Theorem 7 of [29]. See also Theorem 5.7 of [27]). *Let (M, g) be a compact n-dimensional Riemannian manifold having diameter d and with Ricci curvature satisfying $R_{ij} \geq -\kappa(n-1)g_{ij}$ for some constant $\kappa \geq 0$. The first eigenvalue λ_1 of the Laplacian satisfies*

$$\lambda_1 \geq \tfrac{2(1+\kappa(n-1)^2d^2)^{1/2}}{(n+1)d^2} e^{-1-(1+\kappa(n-1)^2d^2)^{1/2}} \geq \tfrac{C_1(n)}{d^2} e^{-C_2(n)\sqrt{\kappa}d} \tag{9.16}$$

where $C_1(n)$ and $C_2(n)$ are constants depending only on the dimension n.

Proof. Suppose $\Delta u = \lambda u$. Note that $\lambda < 0$ and u must change sign on M. Since M is compact, u attains a maximum and minimum on M. Fix a real number a such that $a > -\min u$, so that $u + a > 0$. Let $v = \log(u + a)$. Let $p \in M$ be a point where $|dv|^2$ attains its maximum on M. By (4.54) of Corollary 4.21, at p there holds

$$0 \geq \tfrac{n-1}{2}\Delta|dv|^2 \geq \|dv|^4 - \kappa(n-1)^{-2}|dv|^2 + \lambda\left(a(n+1)e^{-v} - 2\right)|dv|^2 + \lambda^2(ae^{-v}-1)^2. \tag{9.17}$$

Hence, at p,

$$\left(\kappa(n-1)^{-2} + 2\lambda - a\lambda(n+1)e^{-v}\right)|dv|^2 \geq |dv|^4 + \lambda^2(ae^{-v}-1)^2 \geq |dv|^4, \tag{9.18}$$

so that, at p,

$$|dv|^2 \leq \kappa(n-1)^{-2} + 2\lambda - a\lambda(n+1)e^{-v} \leq \kappa(n-1)^{-2} - a\lambda(n+1)e^{-v}, \tag{9.19}$$

where it should be kept in mind that λ is negative.

Let γ be a minimal geodesic running from a point where $|dv|^2$ assumes its minimum to a point where $|dv|^2$ assumes its maximum. On the one hand

$$\int_M |dv|_\gamma = \int_\gamma |d\log(u+a)| \geq \left|\int_\gamma d\log(u+a)\right| = \left|\log\left(\tfrac{a+\max u}{a+\min u}\right)\right| \geq \left|\log\left(\tfrac{a}{a+\min u}\right)\right|. \tag{9.20}$$

On the other hand, by (9.19),

$$\int_M |dv|_\gamma \leq \int_\gamma \left(\kappa(n-1)^{-2} - \tfrac{a\lambda(n+1)}{u+a}\right)^{1/2} \leq d\left(\kappa(n-1)^{-2} - \tfrac{a\lambda(n+1)}{a+\min u}\right)^{1/2}. \tag{9.21}$$

Combining (9.20) and (9.21) shows

$$d^2\left(\kappa(n-1)^{-2} - \tfrac{a\lambda(n+1)}{a+\min u}\right) \geq \left(\log\left(\tfrac{a}{a+\min u}\right)\right)^2, \tag{9.22}$$

so that

$$-\lambda(n+1) \geq \left(d^{-2}\left(\log\left(\tfrac{a}{a+\min u}\right)\right)^2 - \kappa(n-1)^2\right)\tfrac{a+\min u}{a}. \tag{9.23}$$

Writing $t = a^{-1}(a + \min u)$, this takes the form

$$-\lambda(n+1) \geq t\left(d^{-2}(\log t)^2 - \kappa(n-1)^2\right). \tag{9.24}$$

The function $m(t) = t\left(d^{-2}(\log t)^2 - \kappa(n-1)^2\right)$ is maximized when

$$t = e^{-1-\left(\kappa(n-1)^2d^2+1\right)^{1/2}}. \tag{9.25}$$

Sustituting this in (9.24) and simplifying the result yields

$$\begin{aligned} -\lambda(n+1) &\geq e^{-1-\left(\kappa(n-1)^2d^2+1\right)^{1/2}}\left(d^{-2}\left(1+\left(\kappa(n-1)^2d^2+1\right)^{1/2}\right)^2-\kappa(n-1)^2\right) \\ &= \tfrac{2\left(\kappa(n-1)^2d^2+1\right)^{1/2}}{d^2}e^{-1-\left(\kappa(n-1)^2d^2+1\right)^{1/2}}. \end{aligned} \tag{9.26}$$

This shows the first inequality of (9.16). For the second inequality of (9.16). For $x \geq 0$ there holds $(1+x^2)^{1/2} \leq 1 + \frac{1}{2}x^2$, so that

$$\left(\kappa(n-1)^2d^2+1\right)^{1/2} \leq 1 + \tfrac{1}{2}\sqrt{\kappa}(n-1)d. \tag{9.27}$$

Hence

$$\tfrac{2\left(\kappa(n-1)^2d^2+1\right)^{1/2}}{(n+1)d^2}e^{-1-\left(\kappa(n-1)^2d^2+1\right)^{1/2}} \geq \tfrac{2}{(n+1)d^2}e^{-2-\frac{1}{2}(n-1)\sqrt{\kappa}d} \geq \tfrac{C_1(n)}{d^2}e^{-C_2(n)\sqrt{\kappa}d}. \tag{9.28}$$

This shows the second inequality of (9.16). □

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