Geometric Structures Modeled on Affine Hypersurfaces and Generalizations of the Einstein–Weyl and Affine Sphere Equations

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Affine hypersurface structures (*AH structures*) simultaneously generalize Weyl structures and abstract geometric structures induced on a nondegenerate co-oriented hypersurface in flat affine space. The aim of this note is to define equations for AH structures, called *Einstein* which, for Weyl structures, specialize to the usual Einstein Weyl equations, and, in the case of the AH structure induced on a hypersurface in flat affine space, recover the equations for affine spheres. Additionally, we indicate the simplest constructions of Einstein AH structures that do not arise in either of these manners.

A Weyl structure on a n-manifold M comprises a conformal structure $[h]$ and a torsion-free connection ∇ such that $\nabla_i H_{jk} = 0$, where $[h]$ is identified with the weighted tensor $H_{ij} = |\det h|^{-1/n} h_{ij}$. Equivalently, for every $h \in [h]$, there is a one-form $\gamma_i \in \Gamma(T^*M)$ such that $\nabla_i h_{jk} = 2\gamma_i h_{jk}$. When $n > 2$, the Einstein equations for a Weyl structure demand that the symmetric trace-free Ricci tensor $R_{(ij)} - \frac{1}{n} R h_{ij}$ vanishes, where $R = h^{ij} R_{ij}$ and h^{ij} is the inverse of h_{ij}. Note that the Ricci tensor need not be symmetric. Its skew symmetric part is given by $2R_{[ij]} = -nF_{ij} = nd\gamma_{ij}$ and does not depend on the choice of $h \in [h]$. When $n = 2$ the Einstein–Weyl equations, like the usual metric Einstein equations, are vacuous. For the usual Einstein equations the traced differential Bianchi identity implies that the scalar curvature is constant, and constant scalar curvature can be regarded as the 2-dimensional analogue of the Einstein condition. Likewise, for the Einstein–Weyl equations, it follows from the Bianchi identities that $\nabla_i R + n\nabla^p F_{ip} = 0$, where indices are raised and lowered using H_{ij} and its inverse H^{ij}, and where $R = H^{ij} R_{ij}$. Calderbank [1, 2] proposed this equation as the definition of Einstein–Weyl structures in two dimensions, and constructed solutions (see also [4]). This

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point of view was important for identifying an appropriate notion of Einstein AH structure.

Formally, AH structures can be defined by relaxing the compatibility condition defining Weyl structures. First, however, it is convenient to change the perspective slightly. Let the pair $([\nabla], [h])$ comprise a projective structure, $[\nabla]$, meaning an equivalence class of torsion-free connections having the same unparameterized geodesics, and a conformal structure $[h]$. There is a unique *aligned* representative $\nabla \in [\nabla]$ distinguished by the requirement that $\nabla_i H_{jk}$ be completely trace-free. The pair $([\nabla], [h])$ is an AH structure if $\nabla_i H_{jk}$ is completely symmetric, that is $\nabla_{[i} H_{j]k} = 0$. Equivalently, for every $h \in [h]$, there is $\gamma_i \in \Gamma(T^*M)$ such that $\nabla_{[i} h_{j]k} = \gamma_{[i} h_{j]k}$. The *cubic torsion* of the AH structure is $\mathcal{L}_{ij}{}^k = H^{kp} \nabla_i H_{jp}$. The connection $\bar{\nabla} = \nabla + \mathcal{L}_{ij}{}^k$ is the aligned representative of the *conjugate* AH structure $([\bar{\nabla}], [h])$. Its cubic torsion is $-\mathcal{L}_{ij}{}^k$. Conjugacy is an involution on the space of AH structures and its fixed points are exactly Weyl structures.

The second fundamental form Π of an immersed hypersurface M in an $(n+1)$-manifold N with connection $\mathbb{D}$ is the symmetric normal-bundle valued tensor defined by taking $\Pi(X, Y)$ to be the projection onto the normal bundle of $\mathbb{D}_X Y$, where X and Y are tangent to M. Since the difference tensor of projectively equivalent connections has the form $2\sigma_{(i}\delta_{j)}{}^k$ for some $\sigma_i \in \Gamma(T^*M)$, Π depends only on the projective equivalence class $[\mathbb{D}]$, and not on $\mathbb{D}$ itself. The immersion is *nondegenerate* if Π is. When the target $(N, \mathbb{D})$ is a flat affine space, an equivalent condition is for the Gauss map to the projectivization of the dual vector space, assigning to $p \in M$ the annihilator of the tangent space $T_p M$, to be an immersion. In this case the pullback via the Gauss map of the flat projective structure on projective space yields a flat projective structure $[\bar{\nabla}]$ on M. Together with a co-orientation of M, meaning an orientation of its normal bundle, the second fundamental form determines a conformal structure on M. A vector field W transverse to M determines an induced connection ∇, a metric h_{ij} representing Π, a shape operator $S_i{}^j$, and a one-form τ_i by the usual formulas, $\mathbb{D}_X Y = \nabla_X Y + h(X, Y)W$ and $\mathbb{D}_X W = -S(X) + \tau(X)W$. Here, X and Y are tangential to M and $\nabla_X Y$ and $-S(X)$ are the tangential parts of $\mathbb{D}_X Y$ and $\mathbb{D}_X W$, respectively. With $\tilde{W} + f(W + Z)$ in place of W, the induced $\tilde{\nabla}$ and $\tilde{h}$ are given by $\tilde{\nabla}_X Y = \nabla_X Y - h(X, Y)Z$, and $\tilde{h}_{ij} = f^{-1} h_{ij}$. In particular, h generates the conformal structure induced by Π and the given co-orientation. Allowing projective changes of $\mathbb{D}$ and arbitrary changes of W, the equivalence class $\{\nabla\} = \{\nabla + 2\alpha_{(i}\delta_{j)}{}^k - \beta^k h_{ij}\}$ generated by any induced connection depends neither on the choice of transversal X nor on the choice of $\mathbb{D}$ within its projective equivalence class. This class $\{\nabla\}$ is the *conformal projective* equivalence class of ∇. Observe that $2\nabla_{[i} h_{j]k} = -2\tau_{[i} h_{j]k}$, so that any induced connection ∇ generates with $[h]$ an AH structure. The pair $(\{\nabla\}, [h])$ is called a *Codazzi projective structure*, and should be viewed as a generalization of the notion of conformal structure. In this analogy, the AH structures generating a Codazzi projective structure correspond to the individual metrics representing a conformal structure; in particular, the difference tensor of the aligned representatives of two AH structures generating the same conformal projective structure has the form $2\alpha_{(i}\delta_{j)}{}^k - h_{ij} h^{kp} \alpha_p$ of the difference tensor of the Levi–Civita connections of conformal metrics. The induced ∇ is in general not the aligned representative of the AH structure $([\nabla], [h])$ it generates. In fact, there is a unique choice of transverse direction such that this is the case, and this choice is the *affine normal direction*. A distinguished affine normal vector field can be selected by requiring that the volume density induced by the corresponding metric h coincides with that induced from some volume form on the ambient space parallel with respect to a particular ambient connection $\mathbb{D}$. The AH structure conjugate to the AH structure $([\nabla], [h])$ induced via the affine normal is $([\bar{\nabla}], [h])$, where $[\bar{\nabla}]$ is the flat projective structure induced via the conormal Gauss map. In summary, a nondegenerate hypersurface in a projectively flat space carries a conformal projective structure which admits a distinguished subordinate AH structure, that determined by the affine normal, for which the conjugate AH structure is the projectively flat one induced via the conormal Gauss map.

The curvature $R_{ijkl} = R_{ijk}{}^{p}H_{pl}$ of an AH structure means the curvature of its aligned representative ∇. There are three principles useful in understanding the curvature. It can be decomposed by symmetries, it can be decomposed into its self-conjugate and anti-self-conjugate parts, and its pieces can be isolated depending only on the underlying conformal projective equivalence class. These three points of view lead to essentially the same tensors, which are now briefly summarized. There are two possible rank two traces of R_{ijkl}, namely the ordinary Ricci trace $R_{ij} = R_{pij}{}^{p}$, and the trace $R_{ip}{}^{p}{}_{l}$. All further traces lead to a multiple of the weighted scalar curvature $R = H^{ij}R_{ij}$, and the trace-free symmetric Ricci tensor and the trace-free symmetric conjugate Ricci tensor span the space of trace-free rank two traces. The Weyl curvature is the completely trace-free part of R_{ijkl}. It decomposes as $W_{ijkl} = A_{ijkl} + E_{ijkl}$, where the self-conjugate Weyl tensor A_{ijkl} has the symmetries of a metric curvature tensor and the anti-self-conjugate Weyl tensor E_{ijkl} has the symmetries of a symplectic curvature tensor. These two tensors are invariant under conformal projective equivalence. There are corresponding self-conjugate and anti-self-conjugate Cotton tensors, which are invariant, respectively, when A_{ijkl} or E_{ijkl} vanishes. In four dimensions, when the anti-self-conjugate Weyl and Cotton tensors vanish, there is a Bach tensor that directly generalizes the Bach tensor of a Weyl structure. A key role in understanding appropriate generalizations of the Einstein condition is played by the conformal projectively invariant one-form A_i defined by

$$\mathcal{L}^{abc}E_{iabc} = 2(2-n)A_i = (n-2)\left(\nabla^p F_{ip} + \tfrac{1}{n}\nabla_i R + 2\nabla^p\{A\}_{ip} - \mathcal{L}_i{}^{pq}\{W\}_{pq}\right), \tag{1}$$

where the brackets $\{\cdot\}$ indicate the trace-free symmetric part, and A_{ij} and W_{ij} are the self-conjugate and full Schouten tensors, respectively, which are certain linear combinations of the symmetric Ricci and conjugate Ricci tensors and of RH_{ij}. An AH structure is *conservative* if $A_i = 0$. The *naive Einstein* equations require the vanishing of the trace-free symmetric parts of the Ricci and conjugate Ricci tensors. However, these conditions are inadequate to generate (via the Bianchi identities) anything like the constancy of the scalar curvature, and appear too flabby to give rise to a good theory. In the presence of the naive Einstein equation, (1) gives

$\mathcal{L}^{abc}E_{iabc} = 2(2-n)A_i = (n-2)(\nabla^p F_{ip} + \frac{1}{n}\nabla_i R)$. The vanishing of this expression is a consequence of the Einstein–Weyl equations that can be regarded as generalizing constancy of the scalar curvature, and was shown by Calderbank to give a good notion of Einstein–Weyl equations in two dimensions. Coupled with the conformal projective invariance of A_i, this suggests defining an AH structure to be *Einstein* if it is naive Einstein and conservative. By (1), an AH structure with vanishing anti-self-conjugate Weyl tensor is conservative. Coupled with the possibility of constructing a good generalization of the Bach tensor when the anti-self-conjugate Weyl and Cotton tensors vanish, this suggests considering the stronger conditions of the naive Einstein equations plus the vanishing of the anti-self-conjugate Weyl tensor, and possibly also of the anti-self-conjugate Cotton tensor. While it is particularly interesting to construct Einstein AH structures satisfying these stronger conditions, it is not yet clear to what extent they should be regarded as part of the Einstein condition.

By definition, an AH structure is Einstein if and only if its conjugate is Einstein. An affine hypersurface is an *affine sphere* if its affine normals meet in a point or are all parallel. Equivalently, its shape operator is a multiple of the identity. The AH structures induced on a nondegenerate affine hypersurface are Einstein if and only if the hypersurface is an affine sphere. Since the anti-self-conjugate Weyl and Cotton tensors vanish for a conjugate projectively flat AH structure such as that induced via the affine normal, these AH structures automatically satisfy the stronger conditions discussed in the previous paragraph. By a theorem of Cheng and Yau, the interior of a sharp convex cone is foliated in a unique way by hyperbolic affine spheres asymptotic to the boundary of the cone. When this theorem is applied to the cone over the universal cover of a convex flat real projective manifold M, the equiaffine metrics of the affine spheres yield a canonical homothety class of metrics which, together with the given flat projective structure, generates an Einstein AH structure on M. Since convex flat real projective manifolds abound (see [5]), this provides many Einstein AH structures. If these were the only examples of Einstein AH structures there would be no point in introducing the formalism described here. The simplest example of an Einstein AH structure that is neither Weyl nor projectively nor conjugate projectively flat is the following. Let $G = SU(n)$ and define on the Lie algebra $\mathfrak{su}(n)$, regarded as skew-Hermitian matrices, the one-parameter family of commutative, nonassociative multiplications

$$X \circ Y = t\mathrm{i}(XY + YX - \tfrac{2}{n}\,\mathrm{tr}(XY)I).$$

Let h be the bi-invariant Riemannian metric determined by the negative of the Killing form, let D be its Levi–Civita connection, and define a bi-invariant torsion-free connection ∇ by $\nabla_X Y = D_X Y + \frac{1}{2}X \circ Y$. Then $([\nabla], [h])$ is an Einstein AH structure with self-conjugate Weyl and Cotton tensors. Many exact Einstein AH structures $([\nabla], [h])$, for which a representative metric $h \in [h]$ having vanishing γ is flat, can be constructed as follows. Let D be the Levi–Civita connection of the Euclidean metric h on $\mathbb{R}^n$, and let P be a harmonic homogeneous cubic polynomial such that the square of the norm of the Hessian of P is a nonzero constant multiple of the quadratic form corresponding to the Euclidean metric. Then $\nabla = D - P_{ijp}P^{kp}$, where P^{ij} is the bivector dual to the Hessian P_{ij} of P, generates with $[h]$ an exact Einstein AH structure with self-conjugate curvature. In all dimensions $n > 3$ there are P for which the resulting Einstein AH structure is neither projectively flat nor conjugate projectively flat. The simplest examples are the following. A Steiner triple system is a collection $\mathscr{B}$ of 3 element subsets of $\bar{n} = \{1, \dots, n\}$ such that every two element subset of $\bar{n}$ is contained in exactly one $B \in \mathscr{B}$. For $I = abc \in \mathscr{B}$ let $x_I = x_a x_b x_c$. For any choice of $\epsilon_I \in \{\pm 1\}$ the polynomial $P_{\mathscr{B},\epsilon}(x) = \sum_{I \in \mathscr{B}} \epsilon_I x_I$ has the desired properties. The simplest nontrivial example is the seven element Fano projective plane $\mathscr{B} = \{123, 145, 167, 246, 257, 347, 356\}$, yielding

$$P(x) = x_1x_2x_3 + x_1x_4x_5 + x_1x_6x_7 + x_2x_4x_6 + x_2x_5x_7 + x_3x_5x_6 + x_3x_4x_7. \tag{2}$$

Other polynomials with the desired properties include the Cartan cubic isoparametric polynomials, the cubic forms defining the multiplications on the Nahm algebras of compact simple Lie algebras (see [6]), and the cubic form of the Griess algebra preserved by the monster finite simple group. While much more can be said, including general structural statements involving conditions on curvatures, it is out of the scope of the present note to do more than introducing the admittedly complicated formalism. The preceding examples show that the formalism admits solutions more general than the Einstein–Weyl structures and affine spheres that motivated it, and the reader is referred to [3] for a preliminary exposition of further developments.

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