Lattice Paths, k-Bonacci Numbers and Riordan Arrays

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Outline

1. Lattice Path - Pascal Matrix - Delannoy Matrix
2. $(a, b, c, d)-$weighted paths
3. The General Case
4. Schröder Matrix
5. Some additional comments.

Lattice Path – Riordan Arrays

D. Merlini, D. G. Rogers, R. Sprugnoli and M. Cecilia Verri. Underdiagonal lattice paths with unrestricted steps. Discrete Appl. Math., 91(1-3): 197–213, 1999.

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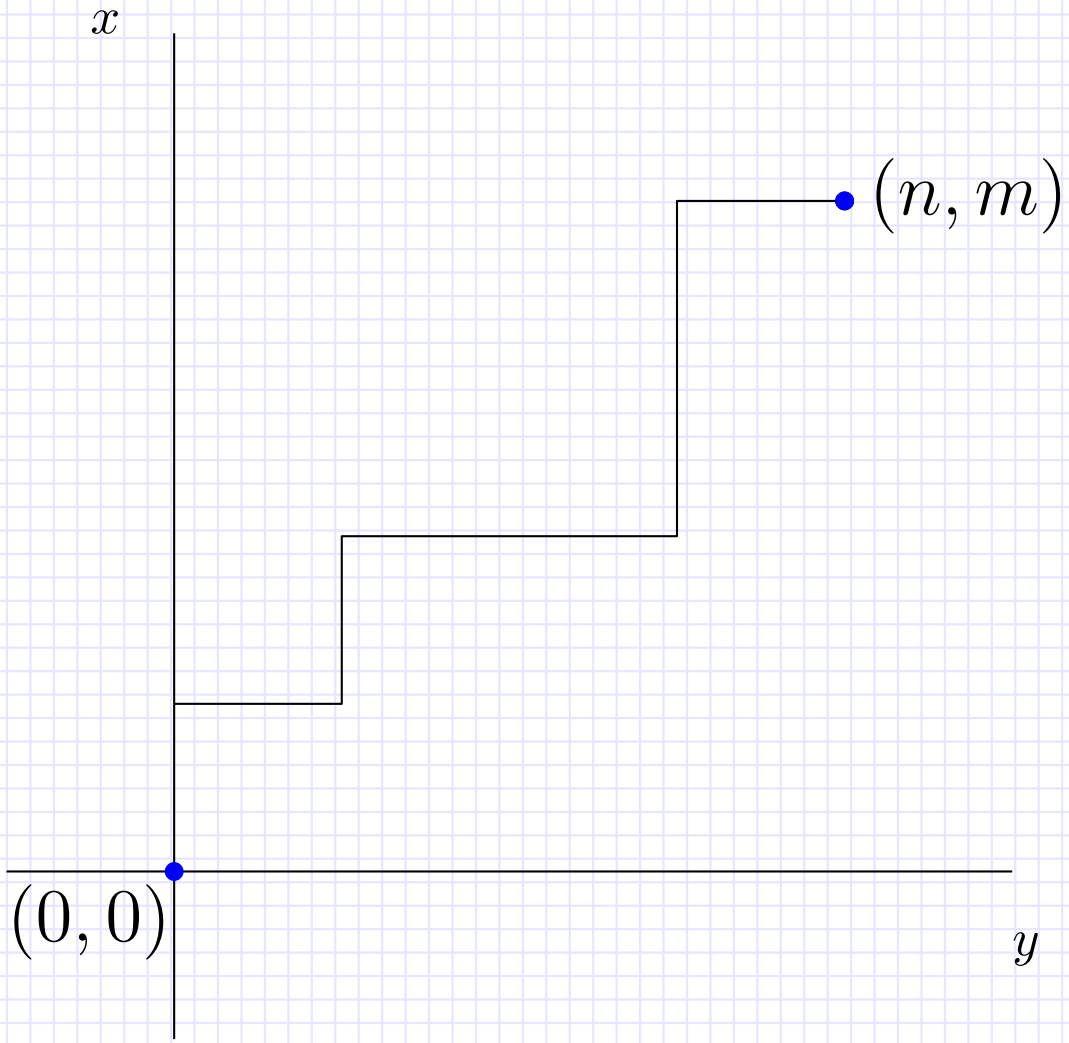
Basic Problem

How many lattice paths are there from (0,0) to (n, m)?



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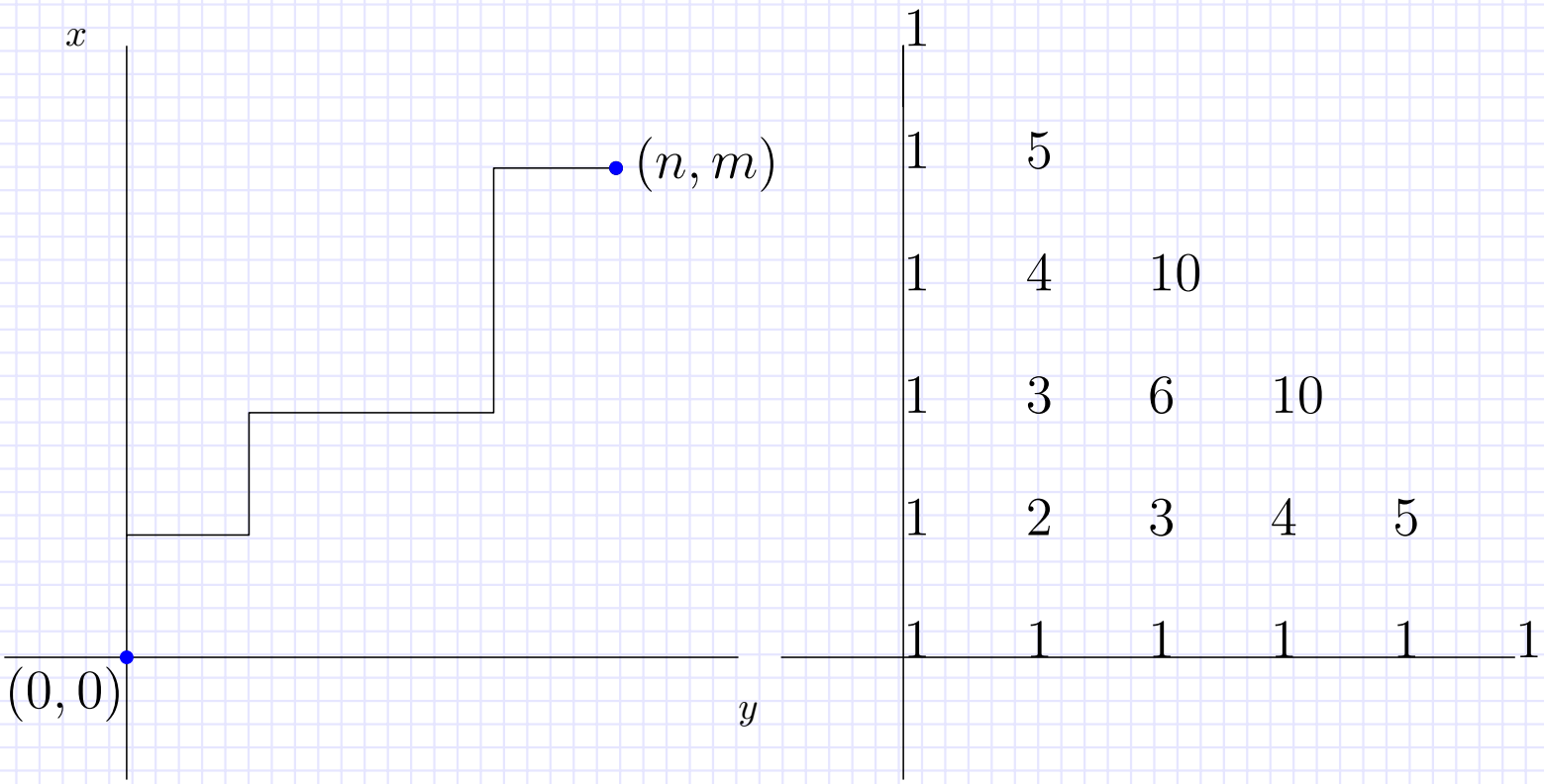


$$\mathcal{D}_1(n, k) = \binom{n+m}{n} = \binom{n+m}{m}$$

Pascal Matrix - Lattice Paths

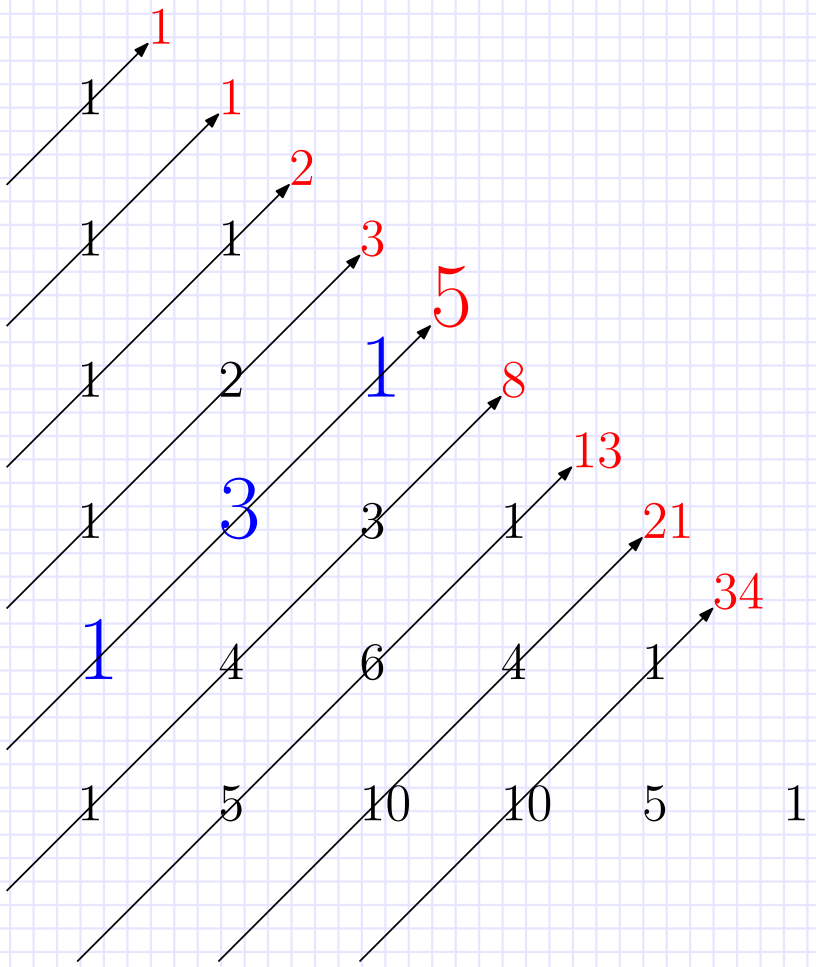
- $\mathcal{H}_1 := \left[h^{(1)}_{n,k}\right]_{n,k\in\mathbb{N}}$, where

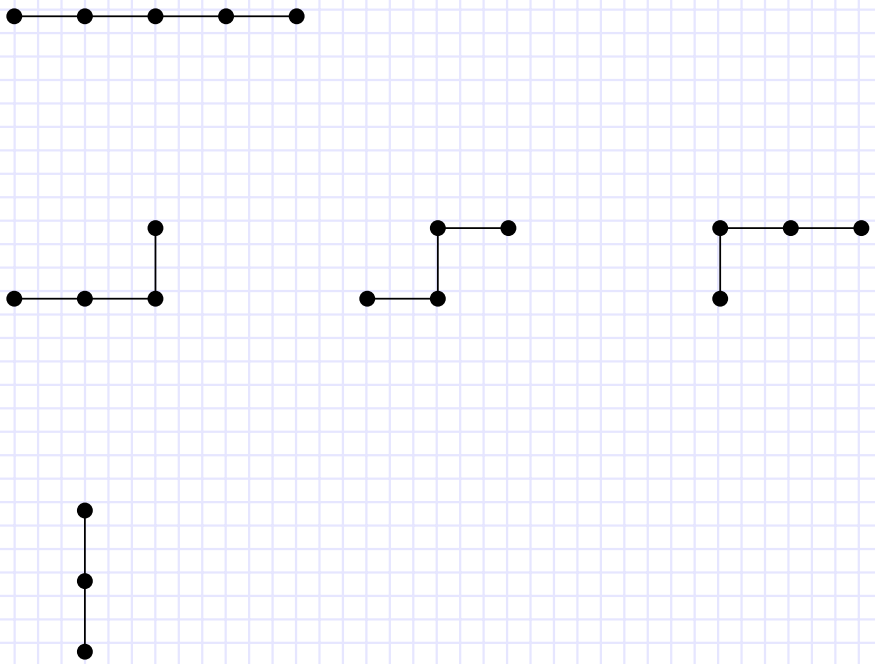
$$h^{(1)}_{n,k} = \begin{cases} \mathcal{D}_1(n-k,k) = \binom{n}{k}, & \text{if } n \geqslant k; \\ 0, & \text{if } n < k. \end{cases}$$

$$\mathcal{H}_1 = \left(\frac{1}{1-z}, \frac{z}{1-z}\right) = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ 1 & 3 & 3 & 1 & & \\ 1 & 4 & 6 & 4 & 1 & \\ 1 & 5 & 10 & 10 & 5 & 1 \\ & & \vdots & & & \end{bmatrix}$$

$\mathcal{H}_1$ **is the Pascal matrix.**

Pascal Matrix - Lattice Paths

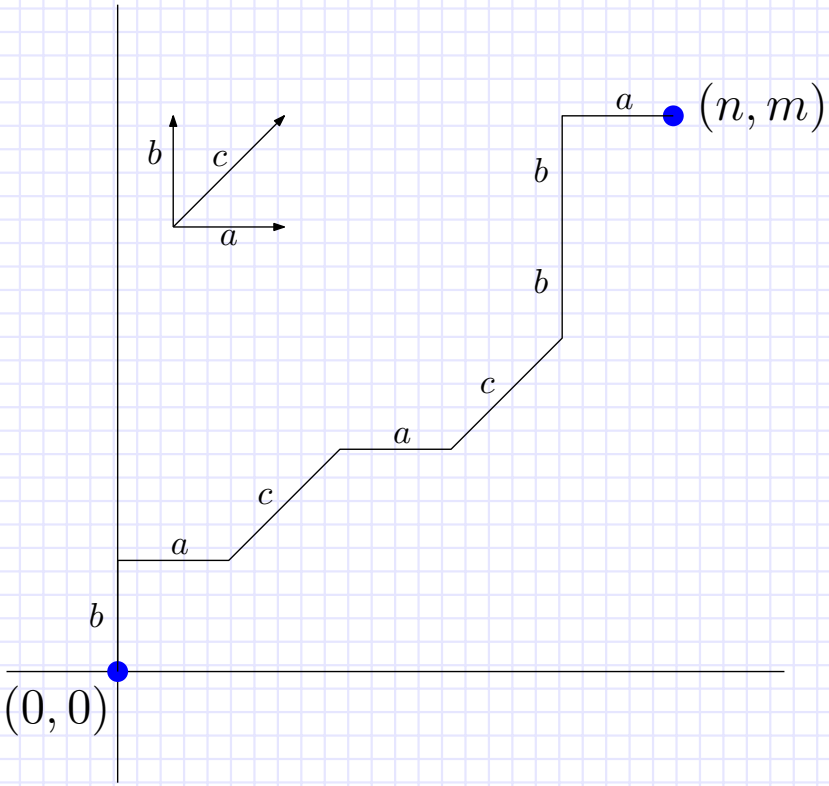


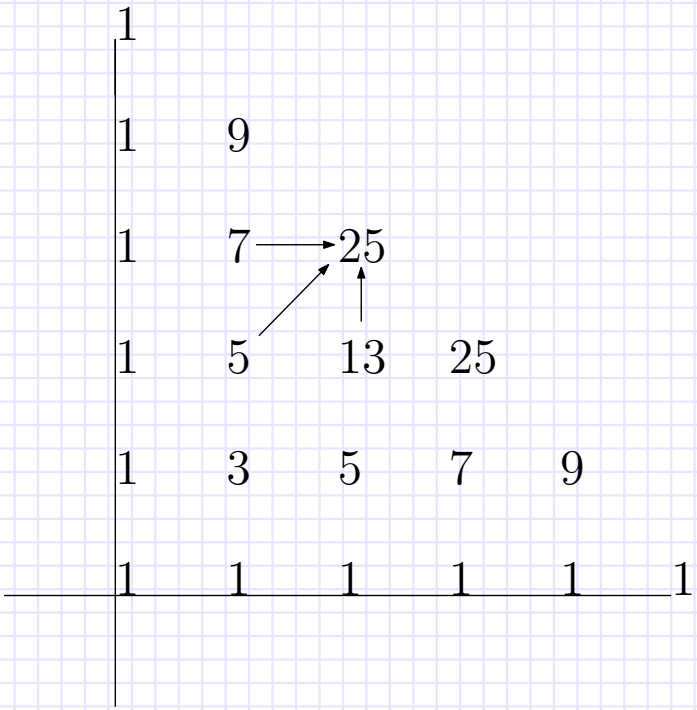


$F_n = \sum_{\ell=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-\ell-1}{\ell} :=$ is the number of lattice path from $(0,0)$ to $(n-1-2k, k)$, for $k = 0, 1, \ldots, \lfloor (n-1)/2 \rfloor$.

Problem 2

How many (a, b, c)-weighted lattice paths are there from (0,0) to (n, m)?





$$\mathcal{D}_2^*(n,k) = \sum_{i=0}^{k} \binom{k}{i} \binom{n+k-i}{k} a^{k-i} b^{n-i} c^i$$

Generalized Delannoy number

Generalized Delannoy matrix: $\mathcal{H}_2(a,b,c) = [d_{n,k}]_{n,k\in\mathbb{N}}$, where

$$d_{n,k} = \begin{cases} \mathcal{D}_2^*(n-k,k), & \text{if } n \geqslant k; \\ 0, & \text{if } n < k. \end{cases}$$

$$\mathcal{H}_2(a,b,c) = \left(\frac{1}{1-bz}, z\frac{a+cz}{1-bz}\right)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \\ b & a & 0 & 0 & 0 & \\ b^2 & c+2ab & a^2 & 0 & 0 & \\ b^3 & b(2c+3ab) & a(2c+3ab) & a^3 & 0 & \\ b^4 & b^2(3c+4ab) & c^2+6abc+6a^2b^2 & a^2(3c+4ab) & a^4 & \cdots \\ \vdots & & \vdots & & \vdots & \end{bmatrix}$$

GF Row Sum: $\frac{1}{1-(a+b)z-cz^2}$

G.-S. Cheon, H. Kim and L. W. Shapiro. A generalization of Lucas polynomial sequence. Discrete Appl. Math. 157(5), 2009.

Delannoy Matrix

Delannoy Matrix:

$$\mathcal{H}_2(1,1,1) = \left(\frac{1}{1-z}, \frac{z(1+z)}{1-z}\right)$$

$$= \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 3 & 1 & & & \\ 1 & 5 & 5 & 1 & & \\ 1 & 7 & 13 & 7 & 1 & \\ 1 & 9 & 25 & 25 & 9 & 1 \\ & & \vdots & & & \end{bmatrix}$$

GF Row Sum:

$$\frac{1}{1-2z-z^2}$$

Delannoy Matrix

The sum of the elements on the rising diagonal is the **tribonacci sequence** t_n.

1					
1	1				
1	1	2			
1	3	4	7		
1	5	5	1	13	
1	7	13	7	1	
1	9	25	25	9	1

Delannoy Matrix

$\mathcal{T}^{(a,b,c)}_{n+1,q,r,p}$:= the sum on the diagonal of direction (r,q), i.e.,

$$\mathcal{T}^{(a,b,c)}_{n+1,q,r,p} = \sum_{k=0}^{\left\lfloor \frac{n-p}{q+r} \right\rfloor} d(n-qk, p+rk).$$

Direction $(1,q)$.

Theorem

The sequence $\mathcal{T}_{n,q} := \mathcal{T}^{(a,b,c)}_{n,q,1,0}$ satisfies the following linear recurrence relation

$$\mathcal{T}_{n,q} = b\mathcal{T}_{n-1,q} + a\mathcal{T}_{n-q-1,q} + c\mathcal{T}_{n-q-2,q},$$

with the initial values $\mathcal{T}_{1,q} = 1$, and $\mathcal{T}_{0,q} = \mathcal{T}_{-1,q} = \cdots = \mathcal{T}_{n-q-2,q} = 0$.

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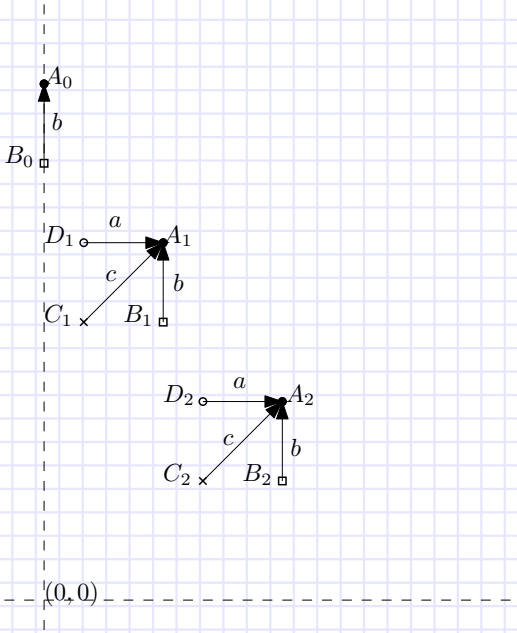
Delannoy Matrix

Proof:



Corollary

The generating function of the sequence $\{\mathcal{T}_{n+1,q}\}_{n\geq 0}$ *is*

$$\sum_{n\geq 0} \mathcal{T}_{n+1,q} z^n = \frac{1}{1 - bz - az^{q+1} + cz^{q+2}}.$$

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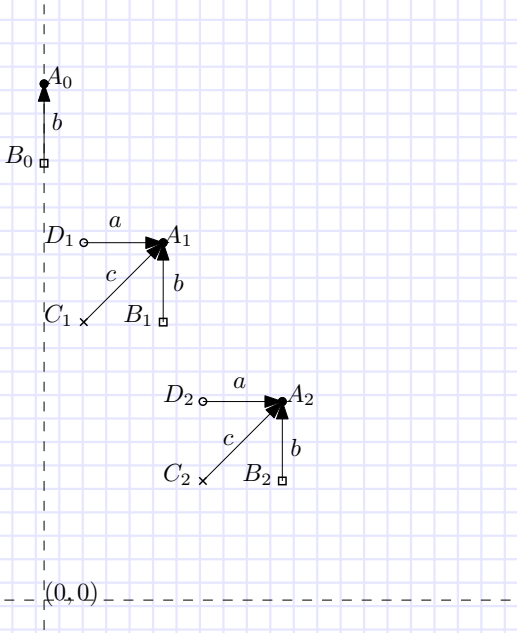

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Delannoy Matrix

General Case: this corresponds to the finite sequences lying over finite transversals of Delannoy matrix.

Theorem

The sequence $\mathcal{T}_{n,q,r,p} := \mathcal{T}^{(a,b,c)}_{n,q,r,p}$ satisfies the following linear recurrence relation

$$\sum_{s=0}^{r}(-b)^s\binom{r}{s}\mathcal{T}_{n-s,q,r,p} = \sum_{s=0}^{r}a^{r-s}c^s\binom{r}{s}\mathcal{T}_{n-r-q-s,q,r,p}.$$

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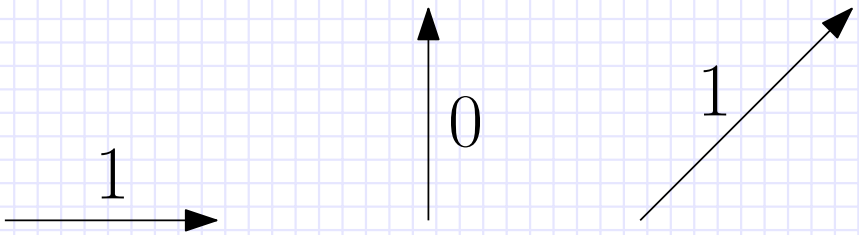
Delannoy Matrix - Tribonacci case

Example

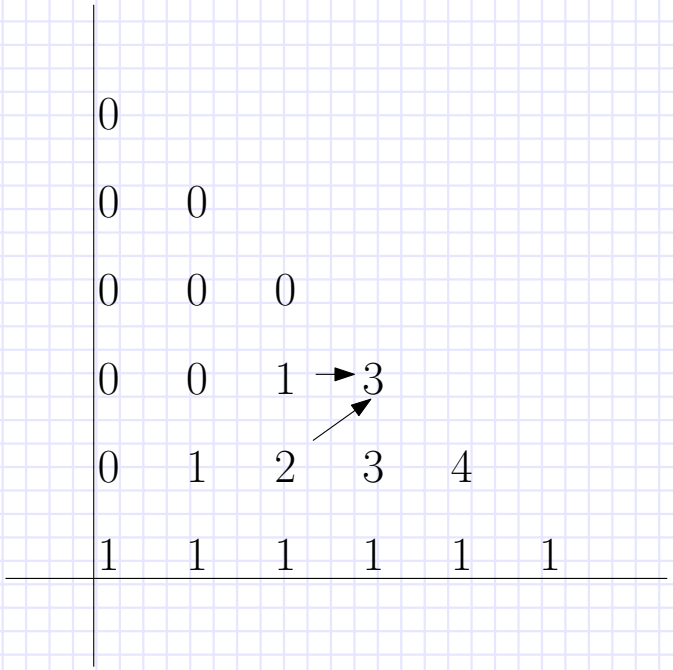
For $r = q = 1$ and $p = 0$, we have the tribonacci sequence $(T_n)_{n\geq 0} = (0, 1, 1, 2, 4, 7, 13, 24, \ldots)$. Therefore, we have the explicit relation

$$T_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{j=0}^{k} \binom{j}{k} \binom{n-j-k}{k}.$$

Particular Case



$$\mathcal{H}_2(1,0,1) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & 0 & \\ 0 & 1 & 1 & 0 & 0 & \\ 0 & 0 & 2 & 1 & 0 & \\ 0 & 0 & 1 & 3 & 1 & \cdots \\ \vdots & & \vdots & & \vdots & \end{bmatrix}$$



Row Sum = Fibonacci Sequence

GF Row Sum:

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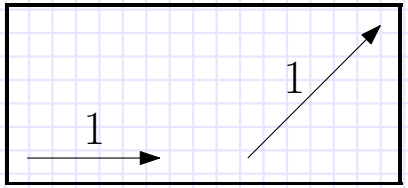
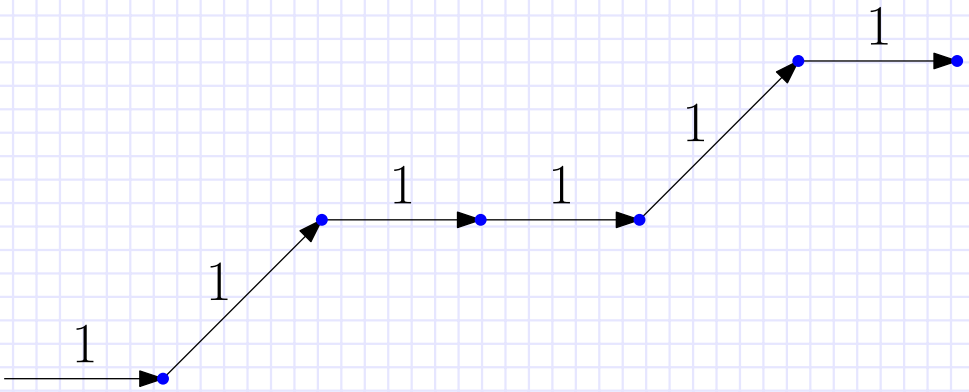
Combinatorial Interpretation - Fibonacci Numbers





$$\boxed{F_{n+1} = \sum_{k=0}^{n} \omega_{k,n-k}}$$

- $\omega_{k,n-k} :=$ is the sum of weights of paths from $(0,0)$ to $(k, n-k)$ using the steps:

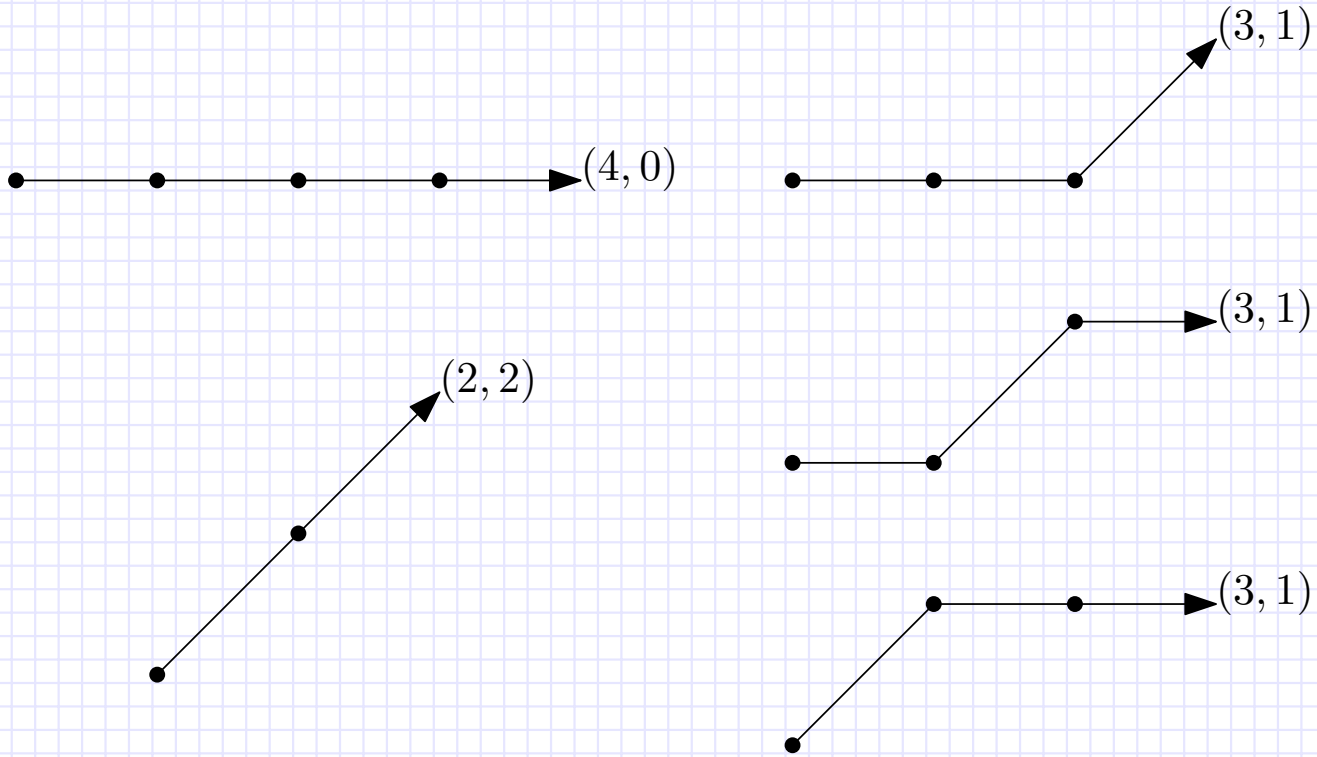
$$(1,0), \qquad (1,1)$$

G.-S. Cheon, H. Kim and L. W. Shapiro. A generalization of Lucas polynomial sequence. Discrete Appl. Math. 157(5), 2009.

Combinatorial Interpretation - Fibonacci Numbers

Example

$F_5 = 5$



Summarizing

- Pascal Matrix – Sum of the elements on the rising diagonal = Fibonacci numbers.
- Delannoy Matrix – Sum of the elements on the rising diagonal = Tribonacci numbers.
- XXXX Matrix (Riordan array) – Sum of the elements on the rising diagonal = k-bonacci numbers.

What family of steps are required to find a Riordan array, such that the sum of the elements on its main diagonal (or on its row) is the k-bonacci sequence $\mathcal{F}_n^{(k)}$?

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Combinatorial Interpretation?

- **k-bonacci numbers** The k-bonacci numbers are defined by the recurrence

$$\boxed{\mathcal{F}_n^{(k)} = \mathcal{F}_{n-1}^{(k)} + \mathcal{F}_{n-2}^{(k)} + \cdots + \mathcal{F}_{n-k}^{(k)}, \quad n \geqslant 1,}$$

  with initial values $\mathcal{F}_{-1}^{(k)} = \mathcal{F}_{-2}^{(k)} = \cdots = \mathcal{F}_{-(k-1)}^{(k)} = 0$ and $\mathcal{F}_0^{(k)} = 1$.

  $k = 3$, tribonacci: $1, 1, 2, 4, 7, 13, 24, \ldots$

- **k-bonacci polynomials**.

$$\boxed{\mathcal{F}_n^{(k)}(x) = x^{k-1}\mathcal{F}_{n-1}^{(k)}(x) + x^{k-2}\mathcal{F}_{n-2}^{(k)} + \cdots + \mathcal{F}_{n-k}^{(k)}, \quad n \geqslant 1,}$$

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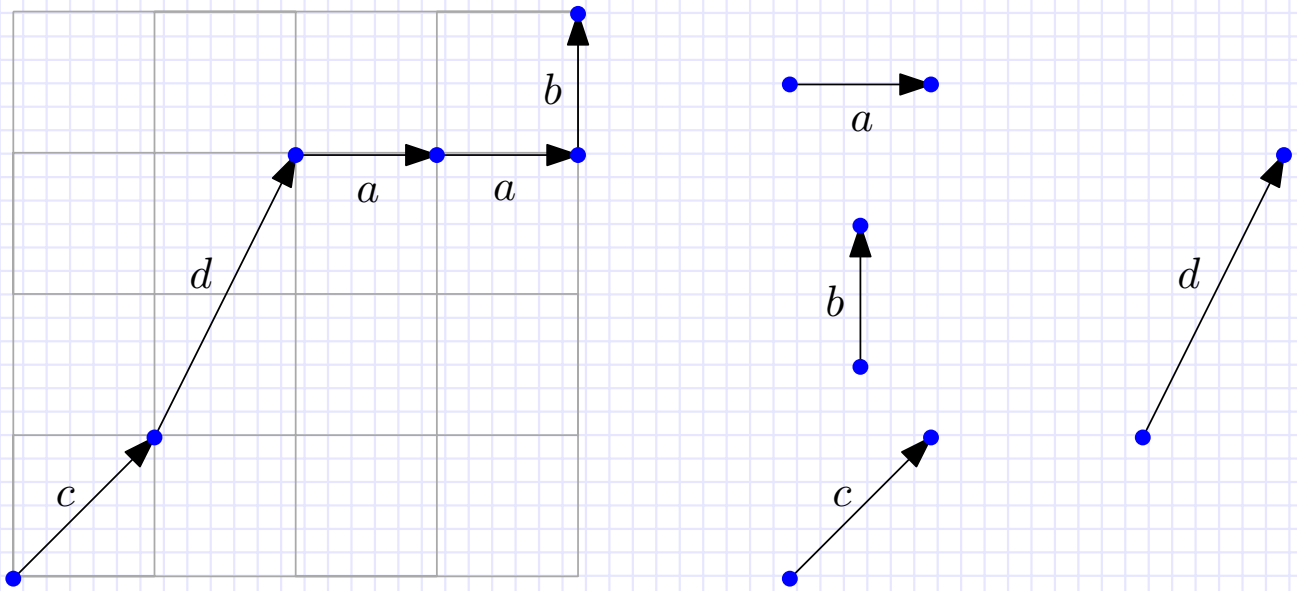
(a, b, c, d)-weighted paths

Let $\mathbb{M}_3(n, k)$ denote the set of (a, b, c, d)-weighted paths from the point $(0, 0)$ to the point (k, n), with step set $S_3 = \{H = (1, 0), V = (0, 1), D_1 = (1, 1), D_2 = (1, 2)\}$, where each step is labelled with weights a, b, c and d, respectively.

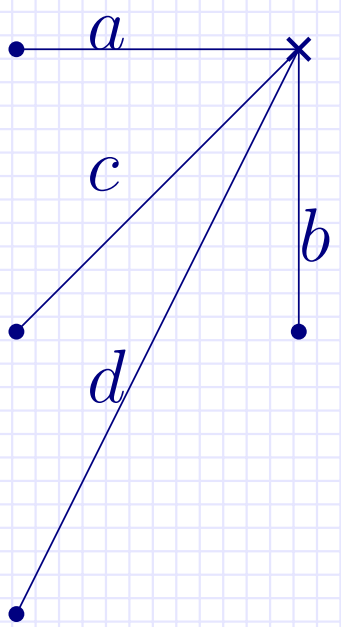
Example



(a, b, c, d)-weighted paths

$$\boxed{\begin{aligned}\mathcal{D}_3(n,k) = a\mathcal{D}_3(n-1,k) + b\mathcal{D}_3(n,k-1) \\ + c\mathcal{D}_3(n-1,k-1) + d\mathcal{D}_3(n-1,k-2),\end{aligned}}$$

with $k \geqslant 2, n \geqslant 1$ and initial conditions $\mathcal{D}_3(0,k) = b^k$ and $\mathcal{D}_3(n,0) = a^n$.



(a, b, c, d)-weighted paths

Theorem

The number of (a, b, c, d)-lattice paths is given by

$$\mathcal{D}_3(n,k) = \sum_{j=0}^{n}\sum_{l=0}^{j}\binom{n}{j}\binom{j}{l}\binom{n+k-2j+l}{k-2j+l}a^{n-j}c^l d^{j-l}b^{k-2j+l}.$$

$$= \sum_{j=0}^{n}\sum_{l=0}^{n-j}\binom{n}{j}\binom{n-j}{l}\binom{k-n+2j+l}{n}a^j c^l d^{n-j-l}b^{k-2n+2j+l}.$$

Proof.

$$\text{For } n \geqslant 1, \text{ let } \quad \mathcal{W}_n^{(3)}(z) := \sum_{i=0}^{\infty}\mathcal{D}_3(n,i)z^i.$$

Then

$$\mathcal{W}_n^{(3)}(z) = a\mathcal{W}_{n-1}^{(3)}(z) + bz\mathcal{W}_n^{(3)}(z) + cz\mathcal{W}_{n-1}^{(3)}(z) + dz^2\mathcal{W}_{n-1}^{(3)}(z).$$

(a, b, c, d)-weighted paths

Proof.

$$\mathcal{W}_n^{(3)}(z) = \left(\frac{a + cz + dz^2}{1 - bz}\right) \mathcal{W}_{n-1}^{(3)}(z) = \left(\frac{a + cz + dz^2}{1 - bz}\right)^n \mathcal{W}_0^{(3)}(z)$$
$$= \left(\frac{a + cz + dz^2}{1 - bz}\right)^n \frac{1}{1 - bz} = \frac{(a + cz + dz^2)^n}{(1 - bz)^{n+1}}.$$

Therefore, by the binomial theorem

$$\left[z^k\right] \mathcal{W}_n^{(3)} = \left[z^k\right] \left((a + cz + dz^2)^n (1 - bz)^{-(n+1)}\right)$$
$$= \sum_{j=0}^{n} \sum_{l=0}^{j} \binom{n}{j} \binom{j}{l} \binom{n + k - 2j + l}{k - 2j + l} a^{n-j} c^l d^{j-l} b^{k-2j+l}.$$

$\square$

(a, b, c, d)-weighted paths

Definition

Let $\mathcal{H}_3 := \mathcal{H}_3(a, b, c, d) := \left[d_{n,k}^{(3)}\right]_{n,k\in\mathbb{N}}$, where

$$d_{n,k}^{(3)} = \begin{cases} \mathcal{D}_3(n-k, k), & \text{if } n \geqslant k; \\ 0, & \text{if } n < k. \end{cases}$$

Theorem

The infinite triangular array $\mathcal{H}_3(a, b, c, d)$ has a Riordan array expression given by

$$\mathcal{H}_3 = \mathcal{H}_3(a, b, c, d) = \left(\frac{1}{1-bz}, z\frac{a+cz+dz^2}{1-bz}\right).$$

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(a, b, c, d)-weighted paths

Proposition

Let $\mathcal{A}_3(z)$ be the generating function for the rows sums of the Riordan array $\mathcal{H}_3$. Then

$$\mathcal{A}_3(z) = \frac{1}{1 - (a+b)z - cz^2 - dz^3}.$$

- Let $a + b = \overline{p}(x), c = \overline{q}(x), d = \overline{r}(x)$.
- Let $\mathcal{F}_n^{(3)}(x)$ be the n-th row sum of $\mathcal{H}_3$.

$$\mathcal{A}_3(z) = \sum_{i=0}^{\infty} \mathcal{F}_i^{(3)}(x) z^i = \frac{1}{1 - \overline{p}(x)z - \overline{q}(x)z^2 - \overline{r}(x)z^3}.$$

If $p = x^2, q = x$ and $r = 1$, we get the tribonacci polynomials.

(a, b, c, d)-weighted paths

Proposition

Let $\mathcal{A}_3(z)$ be the generating function for the rows sums of the Riordan array $\mathcal{H}_3$. Then

$$\mathcal{A}_3(z) = \frac{1}{1-(a+b)z-cz^2-dz^3}.$$

- Let $a+b=\overline{p}(x), c=\overline{q}(x), d=\overline{r}(x)$.
- Let $\mathcal{F}_n^{(3)}(x)$ be the n-th row sum of $\mathcal{H}_3$.

$$\mathcal{A}_3(z) = \sum_{i=0}^{\infty} \mathcal{F}_i^{(3)}(x) z^i = \frac{1}{1-\overline{p}(x)z-\overline{q}(x)z^2-\overline{r}(x)z^3}.$$

If $p=x^2, q=x$ and $r=1$, we get the tribonacci polynomials.

(a, b, c, d)-weighted paths

Theorem

For all integer $n \geq 3$, the polynomials $\mathcal{F}_n^{(3)}(x)$ satisfy the following recurrence:

$$\boxed{\mathcal{F}_n^{(3)}(x) = \overline{p}(x)\mathcal{F}_{n-1}^{(3)}(x) + \overline{q}(x)\mathcal{F}_{n-2}^{(3)}(x) + \overline{r}(x)\mathcal{F}_{n-3}^{(3)}(x),}$$

where $\mathcal{F}_0^{(3)}(x) = 1, \mathcal{F}_1^{(3)}(x) = \overline{p}(x), \mathcal{F}_2^{(3)}(x) = \overline{p}^2(x) + \overline{q}(x)$.

(a, b, c, d)-weighted paths

Theorem

For all integer $n \geq 0$, and for any polynomials $\overline{p}(x), \overline{q}(x), \overline{r}(x)$

$$\mathcal{F}_n^{(3)}(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{i} \binom{n-i-j}{i} \binom{i}{j} \overline{q}^{i-j}(x) \overline{r}^{j}(x) \overline{p}^{n-2i-j}(x).$$

$$\mathcal{F}_n^{(3)}(x) = \sum_{0 \leq j \leq i \leq n} \binom{n-i}{i-j} \binom{i-j}{j} \overline{q}^{i-2j}(x) \overline{r}^{j}(x) \overline{p}^{n-2i+j}(x).$$

(a, b, c, d)-weighted paths - Combinatorial Interpretation

For any polynomial $\mathcal{F}_n^{(3)}(x)$, we have

$$\mathcal{F}_n^{(3)}(x) = \sum_{k=0}^{n} \omega_{n,k}^{(3)}, \ \ n \geq 0,$$

where $\omega_{n,k}^{(3)}(x) := h_{n,k}^{(3)}(x) = \mathcal{D}_3(n-k, k)$ is the sum of weights of $(a(x), b(x), c(x), d(x))$-weighted paths from $(0, 0)$ to $(k, n-k)$ with step set

$$S_3 = \{H = (0, 0), V = (0, 1), D_1 = (1, 1), D_2 = (1, 2)\},$$

such that $a(x) + b(x) = \overline{p}(x), c(x) = \overline{q}(x)$ and $d(x) = \overline{r}(x)$.

Tribonacci paths

The tribonacci polynomials $T_n(x)$:

$$T_n(x) = x^2 T_{n-1}(x) + xT_{n-2}(x) + T_{n-3}(x), \quad \text{for} \quad n \geqslant 2.$$

$$\sum_{n=0}^{\infty} T_n(x) z^n = \frac{1}{1 - x^2 z - xz^2 - z^3}.$$

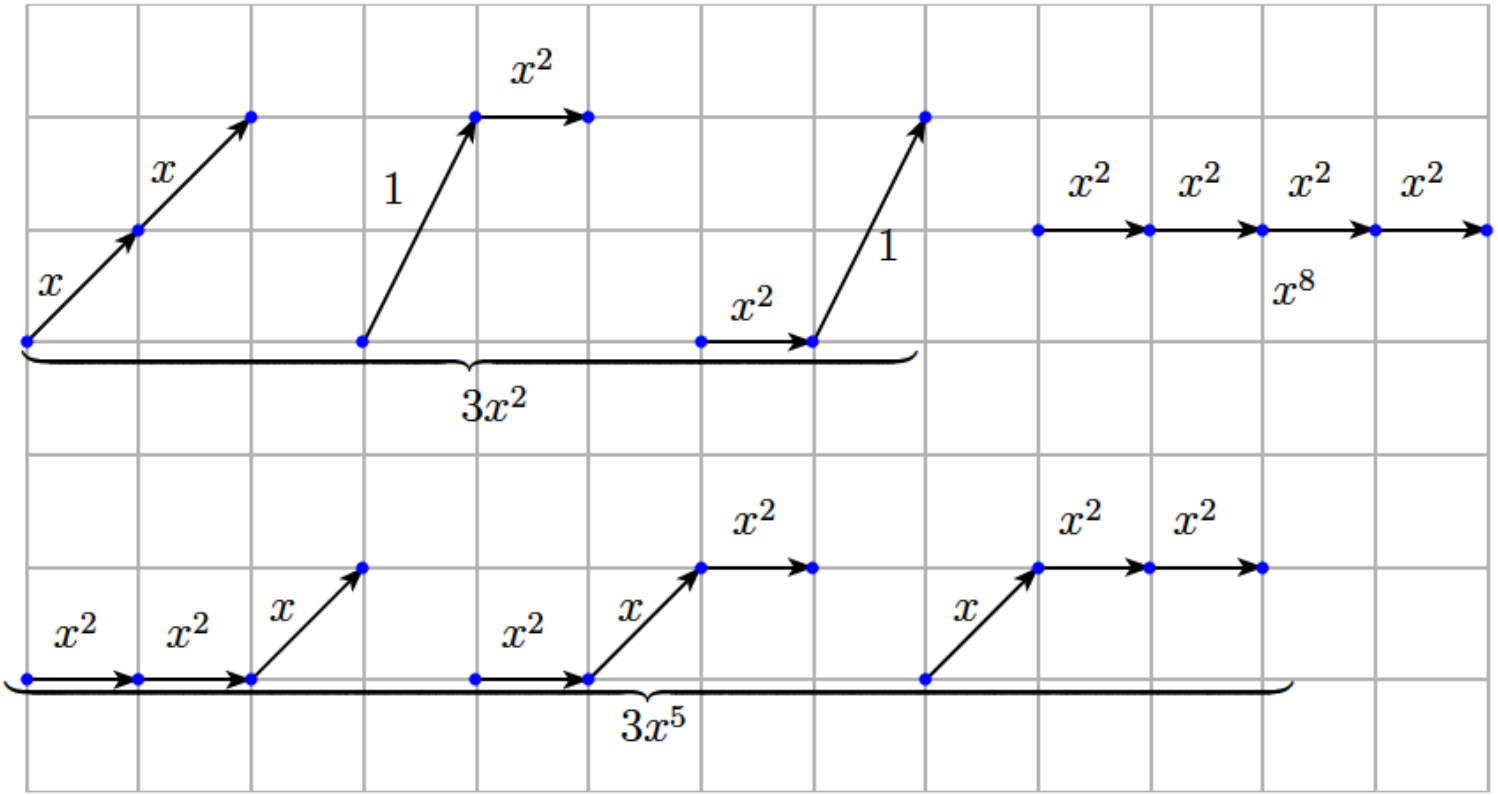
$a(x) = x^2, b(x) = 0, c(x) = x$ and $d(x) = 1$ $\mathcal{H}_3(x^2, 0, x, 1)$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & x^2 & 0 & 0 & 0 & 0 & 0 & \\ 0 & x & x^4 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 2x^3 & x^6 & 0 & 0 & 0 & \\ 0 & 0 & 3x^2 & 3x^5 & x^8 & 0 & 0 & \\ 0 & 0 & 2x & 6x^4 & 4x^7 & x^{10} & 0 & \\ 0 & 0 & 1 & 7x^3 & 10x^6 & 5x^9 & x^{12} & \\ \vdots & & & \vdots & & & & \ddots \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix} = \begin{bmatrix} T_0(x) \\ T_1(x) \\ T_2(x) \\ T_3(x) \\ T_4(x) \\ T_5(x) \\ T_6(x) \\ \vdots \end{bmatrix}.$$

Tribonacci paths

Example

$T_4(x) = x^8 + 3x^5 + 3x^2 = \sum_{i=0}^{4} \omega_{4,i}^{(3)}$. Note that,
$\omega_{0,4}^{(3)} = \omega_{1,3}^{(3)} = 0, \omega_{2,2}^{(3)} = 3x^2, \omega_{3,1}^{(3)} = 3x^5$ and $\omega_{4,0}^{(3)} = x^8$.



Outline

1. Lattice Path - Pascal Matrix - Delannoy Matrix ✓
2. $(a, b, c, d)-$weighted paths ✓
3. The General Case
4. Schröder Matrix
5. Some additional comments

The General Case

- Let $\vec{A}_s = (a_1, a_2, \ldots, a_s)$ be a vector of weights.
- We denote by $\mathbb{M}_m(n,k)$ the set of $\vec{A}_{m+1}$-weighted paths from the point $(0,0)$ to the point (k,n), with step set

$$S_m = \{H = (1,0), V = (0,1), D_1 = (1,1), \ldots, D_{m-1} = (1, m-1)\},$$

  where each step is labelled with weights $a_1, a_2, \ldots, a_{m+1}$, respectively.
- $\mathcal{D}_m(n,k) = |\mathbb{M}_m(n,k)|$.

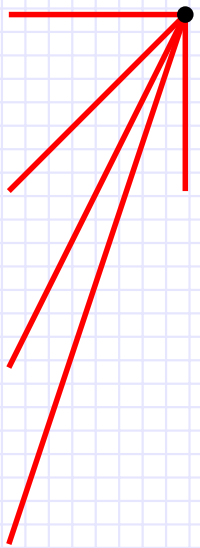
The General Case

Lemma

The numbers $\mathcal{D}_m(n,k)$ satisfy the following $(m+1)$-term recurrence relation

$$\mathcal{D}_m(n,k) = a_1\mathcal{D}_m(n-1,k) + a_2\mathcal{D}_m(n,k-1) + \sum_{j=1}^{m-1} a_{j+2}\mathcal{D}_m(n-1,k-j$$

with $k \geq m-1, n \geq 1$ and initial conditions $\mathcal{D}_m(0,k) = a_2^k$ and $\mathcal{D}_m(n,0) = a_1^n$.



$$S_m = \{H = (1,0), V = (0,1), D_1 = (1,1), \ldots, D_{m-1} = (1, m-1)\},$$

Riordan Arrays and Generalized k-bonacci numbers

Theorem

The number of $\vec{A}_{m+1}$-lattice paths is given by

$$\mathcal{D}_m(n,k) = \sum_{j_1=0}^{n} \sum_{j_2=0}^{n-j_1} \cdots \sum_{j_{m-1}=0}^{n-\sum_{j=1}^{m-2} j_i} \binom{n}{j_1} \binom{n-j_1}{j_2} \cdots \binom{n-\sum_{j=1}^{m-2} j_i}{j_{m-1}}$$
$$\times \binom{n+k-u}{n} a_1^{j_1} a_3^{j_2} \cdots a_m^{j_{m-1}} a_{m+1}^{n-\sum_{i=1}^{m-1} j_i} a_2^{k-u},$$

where

$$u = (m-1)(n-j_1) + \sum_{i=2}^{m-1} (i-m) j_i.$$

Riordan Arrays and Generalized k-bonacci numbers

Definition

Let $\mathcal{H}_m := \mathcal{H}_m(a_1, a_2, \ldots, a_{m+1}) := \left[d_{n,k}^{(m)}\right]_{n,k\in\mathbb{N}}$, where

$$d_{n,k}^{(m)} = \begin{cases} \mathcal{D}_m(n-k,k), & \text{if } n \geqslant k; \\ 0, & \text{if } n < k. \end{cases}$$

Theorem

The infinite triangular array $\mathcal{H}_m$ has a Riordan array expression given by

$$\mathcal{H}_m = \left(\frac{1}{1-a_2z}, z\frac{a_1 + a_3z + a_4z^2 + \cdots + a_{m+1}z^{m-1}}{1-a_2z}\right).$$

Riordan Arrays and Generalized k-bonacci numbers

Proposition

The k-bonacci triangle $\mathcal{T}_k$ defined by the following Riordan array

$$\mathcal{T}_k = \left(\frac{1}{1-z}, z\frac{1+z+\cdots+z^{k-1}}{1-z} \right),$$

satisfies that the sum on the rising diagonal is the k-bonacci sequence $\mathcal{F}_n^{(k)}$, where $\mathcal{F}_n^{(k)} = \mathcal{F}_{n-1}^{(k)} + \mathcal{F}_{n-2}^{(k)} + \cdots + \mathcal{F}_{n-k}^{(k)}$.

Riordan Arrays and Generalized k-bonacci numbers

Proposition

Let $\mathcal{A}_m(z)$ be the generating function for the rows sums of the Riordan array $\mathcal{H}_m$. Then

$$\mathcal{A}_m(z) = \frac{1}{1-(a_1+a_2)z - a_3z^2 - \cdots - a_{m+1}z^m}.$$

Let $\mathcal{F}_n^{(k)}(x)$ be the n-th row sum of $\mathcal{H}_k$ (generalized k-bonacci polynomials):

$$\mathcal{A}_k(z) = \sum_{i=0}^{\infty} \mathcal{F}_i^{(k)}(x)z^i = \frac{1}{1-\overline{p}_1(x)z - \overline{p}_2(x)z^2 - \cdots - \overline{p}_k(x)z^k},$$

where $\overline{p}_1(x) = a_1 + a_2$, and $\overline{p}_j(x) = a_{j+1}, j = 2, \ldots, k$.

Riordan Arrays and Generalized k-bonacci numbers

Proposition

Let $\mathcal{A}_m(z)$ be the generating function for the rows sums of the Riordan array $\mathcal{H}_m$. Then

$$\mathcal{A}_m(z) = \frac{1}{1-(a_1+a_2)z - a_3z^2 - \cdots - a_{m+1}z^m}.$$

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where $\overline{p}_1(x) = a_1 + a_2$, and $\overline{p}_j(x) = a_{j+1}, j = 2, \ldots, k$.

Riordan Arrays and Generalized k-bonacci numbers

Theorem

The polynomials $\mathcal{F}_n^{(k)}(x)$ satisfy the following recurrence for $n \geqslant k$

$$\mathcal{F}_n^{(k)}(x) = \overline{p}_1(x)\mathcal{F}_{n-1}^{(k)}(x) + \overline{p}_2(x)\mathcal{F}_{n-2}^{(k)}(x) + \cdots + \overline{p}_k(x)\mathcal{F}_{n-k}^{(k)}(x),$$

where $\mathcal{F}_0^{(k)}(x) = 1, \mathcal{F}_i^{(k)}(x) = 0$ for $i = -1, -2, \ldots, -(k-1)$.

Combinatorial Interpretation and Generalized k-bonacci numbers

For any polynomial $\mathcal{F}_n^{(m)}(x)$, we have

$$\mathcal{F}_n^{(m)}(x) = \sum_{k=0}^{n} \omega_{n,k}^{(m)}, \ \ n \geq 0,$$

where $\omega_{n,k}^{(m)}(x) := h_{n,k}^{(m)} = \mathcal{D}_m(n-k,k)$ is the sum of weights of $\vec{A}_{m+1}$-weighted paths from $(0,0)$ to $(k, n-k)$ with step set

$$S_m = \{H = (0,0), V = (0,1), D_1 = (1,1), D_2 = (1,2), \\ \ldots, D_{m-1} = (1, m-1)\},$$

such that $a_1(x) + a_2(x) = \overline{p}_1(x)$, and $a_i(x) = \overline{p}_{i-1}(x)$ for $i = 1, \ldots, m+1$.

Riordan Arrays and Generalized k-bonacci numbers

Example

The tetrabonacci numbers:

$$l_0 = 1, \quad l_1 = 1, \quad l_2 = 2, \quad l_3 = 4$$
$$l_{n+4} = l_{n+3} + l_{n+2} + l_{n+1} + l_n, \quad \text{for} \quad n \geqslant 0.$$

The tetrabonacci polynomials $R_n(x)$ are defined by the recurrence relation

$$R_0(x) = 1, \quad R_1(x) = x^3, \quad R_2(x) = x^6 + x^2, \quad R_3(x) = x^9 + 2x^5 + x,$$
$$R_{n+4}(x) = x^3 R_{n+3}(x) + x^2 R_{n+2}(x) + x R_{n+1}(x) + R_n(x), \quad \text{for } n \geqslant 0.$$

If $a_1(x) = x^3, a_2(x) = 0, a_3(x) = x^2, a_4(x) = x$ and $a_5(x) = 1$, we obtain the tetrabonacci polynomials from the row sums of the Riordan array $\mathcal{H}_4(x^3, 0, x^2, x, 1)$.

Riordan Arrays and Generalized k-bonacci numbers

Example

The tetrabonacci numbers:

$$l_0 = 1, \quad l_1 = 1, \quad l_2 = 2, \quad l_3 = 4$$
$$l_{n+4} = l_{n+3} + l_{n+2} + l_{n+1} + l_n, \quad \text{for} \quad n \geqslant 0.$$

The tetrabonacci polynomials $R_n(x)$ are defined by the recurrence relation

$$R_0(x) = 1, \quad R_1(x) = x^3, \quad R_2(x) = x^6 + x^2, \quad R_3(x) = x^9 + 2x^5 + x,$$
$$R_{n+4}(x) = x^3 R_{n+3}(x) + x^2 R_{n+2}(x) + x R_{n+1}(x) + R_n(x), \quad \text{for } n \geqslant 0.$$

If $a_1(x) = x^3, a_2(x) = 0, a_3(x) = x^2, a_4(x) = x$ and $a_5(x) = 1$, we obtain the tetrabonacci polynomials from the row sums of the Riordan array $\mathcal{H}_4(x^3, 0, x^2, x, 1)$.

Riordan Arrays and Generalized k-bonacci numbers

Example

$R_3 = x^9 + 2x^5 + x$

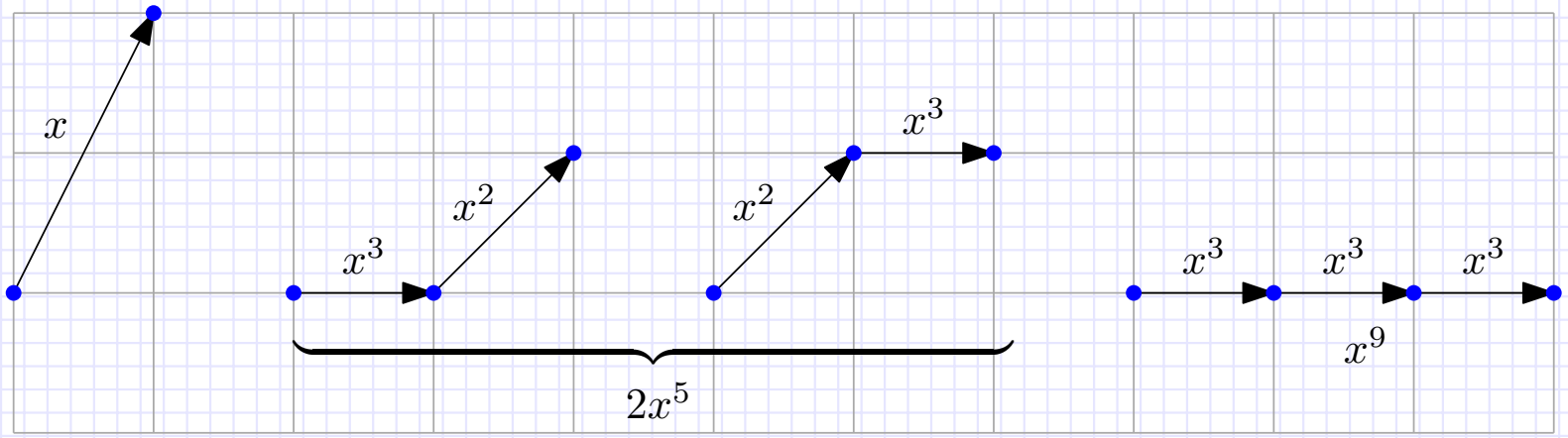

Figure: $(x^3, x^2, 0, x, 1)$-weighted paths and tretranacci polynomials.

J. L. R., V. Sirvent. A generalization of the k-bonacci sequence from Riordan arrays. The Electronic Journal of Combinatorics, 22(1), P1.38, 2015.

Outline

1. Lattice Path - Pascal Matrix - Delannoy Matrix ✓
2. $(a, b, c, d)-$weighted paths ✓
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5. Some additional comments

$$\begin{tikzcd}
\text{Delannoy Matrix} \arrow[r] \arrow[d] & \text{Inverse Matrix} \arrow[d] \\
\text{Combinatorial Interpretation - Lattice Path} &
\end{tikzcd}$$

Schröder matrix as inverse of Delannoy matrix

$$\mathcal{H}_2 = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 3 & 1 & & & \\ 1 & 5 & 5 & 1 & & \\ 1 & 7 & 13 & 7 & 1 & \\ 1 & 9 & 25 & 25 & 9 & 1 \\ & & \vdots & & & \end{bmatrix}$$

Delannoy matrix.

$$\mathcal{H}_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 2 & -3 & 1 & 0 & 0 & 0 \\ -6 & 10 & -5 & 1 & 0 & 0 \\ 22 & -38 & 22 & -7 & 1 & 0 \\ -90 & 158 & -98 & 38 & -9 & 1 \\ & & \vdots & & & \end{bmatrix}$$

Inverse of Delannoy matrix.

Schröder matrix as inverse of Delannoy matrix

$$\mathcal{H}_2 = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 3 & 1 & & & \\ 1 & 5 & 5 & 1 & & \\ 1 & 7 & 14 & 7 & 1 & \\ 1 & 9 & 25 & 25 & 9 & 1 \\ & & \vdots & & & \end{bmatrix}$$

Delannoy matrix.

$$\mathcal{R} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 & 0 \\ 6 & 10 & 5 & 1 & 0 & 0 \\ 22 & 38 & 22 & 7 & 1 & 0 \\ 90 & 158 & 98 & 38 & 9 & 1 \\ & & \vdots & & & \end{bmatrix}$$

Schröder Matrix.

Combinatorial Interpretation?

Schröder matrix as inverse of Delannoy matrix

$$\mathcal{H}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ b & 1 & 0 & 0 & 0 \\ b^2 & 2b+c & 1 & 0 & 0 \\ b^3 & 3b^2+2cb & 3b+2c & 1 & 0 \\ b^4 & 4b^3+3cb^2 & 6b^2+6cb+c^2 & 4b+3c & 1 \\ & & \vdots & & \end{bmatrix}$$

Generalized Delannoy matrix.

$$\mathcal{R} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ b & 1 & 0 & 0 & 0 \\ b^2+cb & 2b+c & 1 & 0 & 0 \\ b^3+3cb^2+2c^2b & 3b^2+5cb+2c^2 & 3b+2c & 1 & 0 \\ & & \vdots & & \end{bmatrix}$$

Generalized Schröder Matrix.

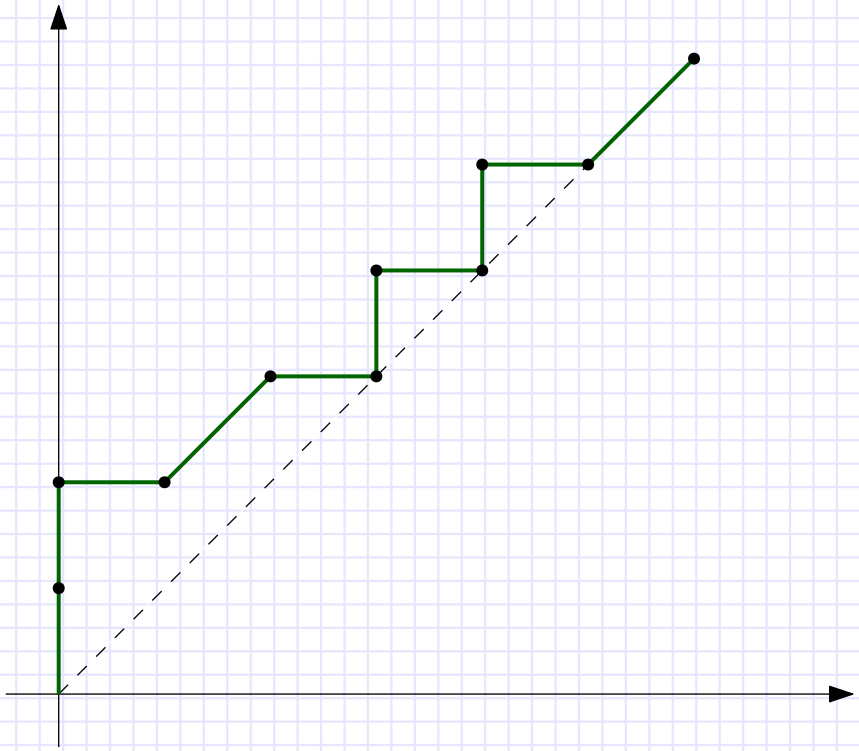
Combinatorial Interpretation?

Schröder matrix

- $\mathbb{J}_3(n,k)$:= denote the set of weighted paths from the point $(0,0)$ to the point $(n-k,n)$, whose step set is $R_3 = \{V=(0,1), D=(1,1), H_1=(1,0)\}$, such that each step is labelled with weights c, d, e_1, respectively; and that its last step is not a horizontal step and that never falling below line $y=x$.
- $\mathcal{L}_3(n,k)$:= total sum of the weights of all weight paths in $\mathbb{J}_3(n,k)$.

S.-L. Yang, S.-N. Zheng, S.-P. Yuan and T.-X. He. Schröder matrix as inverse of Delannoy matrix. Linear Algebra Appl. 439(11): 3605–3614, 2013.

Schröder matrix



Schröder matrix

Generalized Schröder Matrix $\mathcal{S}_3(c, d, e_1) = [\mathcal{L}_3(n, k)]_{n,k\in\mathbb{N}}$,

The unsigned matrix $\mathcal{H}_3^{-1}(1, 1, 1)$ is equal to $\mathcal{S}_3(1, 1, 1)$.

Problem

$$\mathcal{H}_3(a,b,c,d) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ b & 1 & 0 & 0 & 0 \\ b^2 & 2b+c & 1 & 0 & 0 \\ b^3 & 3b^2+2cb+d & 3b+2c & 1 & 0 \\ b^4 & 4b^3+3cb^2+2db & 6b^2+6cb+c^2+2d & 4b+3c & 1 \\ \vdots & & \vdots & & \vdots \end{bmatrix}$$

Generalized Lattice Matrix (4 steps).

$$\mathcal{S}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ b & 1 & 0 & 0 & 0 \\ b^2+cb & 2b+c & 1 & 0 & 0 \\ b^3+3cb^2+2c^2b-db & 3b^2+5cb+2c^2-d & 3b+2c & 1 & 0 \\ & & \vdots & & \end{bmatrix}$$

Generalized Inverse Lattice Matrix (4 Steps).

Combinatorial Interpretation?

Generalized Schröder matrix

- $\mathbb{J}_4(n,k)$:= the family of weighted lattice paths from the point $(0,0)$ to the point $(n-k,n)$, whose step set is

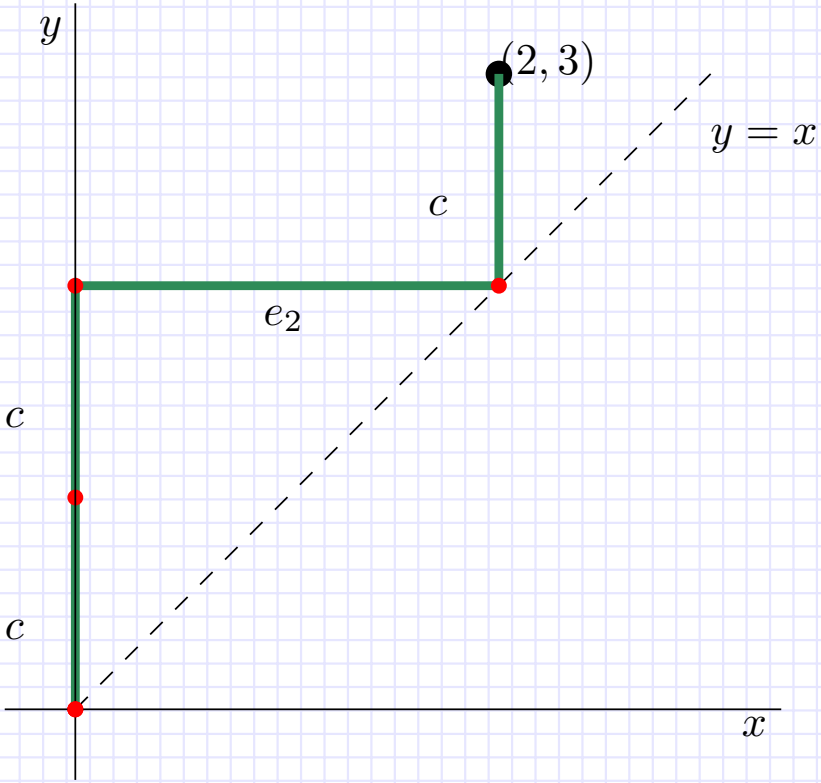
$$R_4 = \{V = (0,1), D = (1,1), H_1 = (1,0), H_2 = (2,0)\},$$

  such that each step is labelled with weights c, d, e_1, e_2, respectively, the path never falls below the line $y = x$, and its last step is not a horizontal step.
- $\mathcal{L}_4(n,k)$:= total sum of the weights of all weight paths in $\mathbb{J}_4(n,k)$.
- $\mathcal{S}_4(c,d,e_1,e_2) := (\mathcal{L}_4(n,k))_{n,k\in\mathbb{N}}$.

Schröder matrix

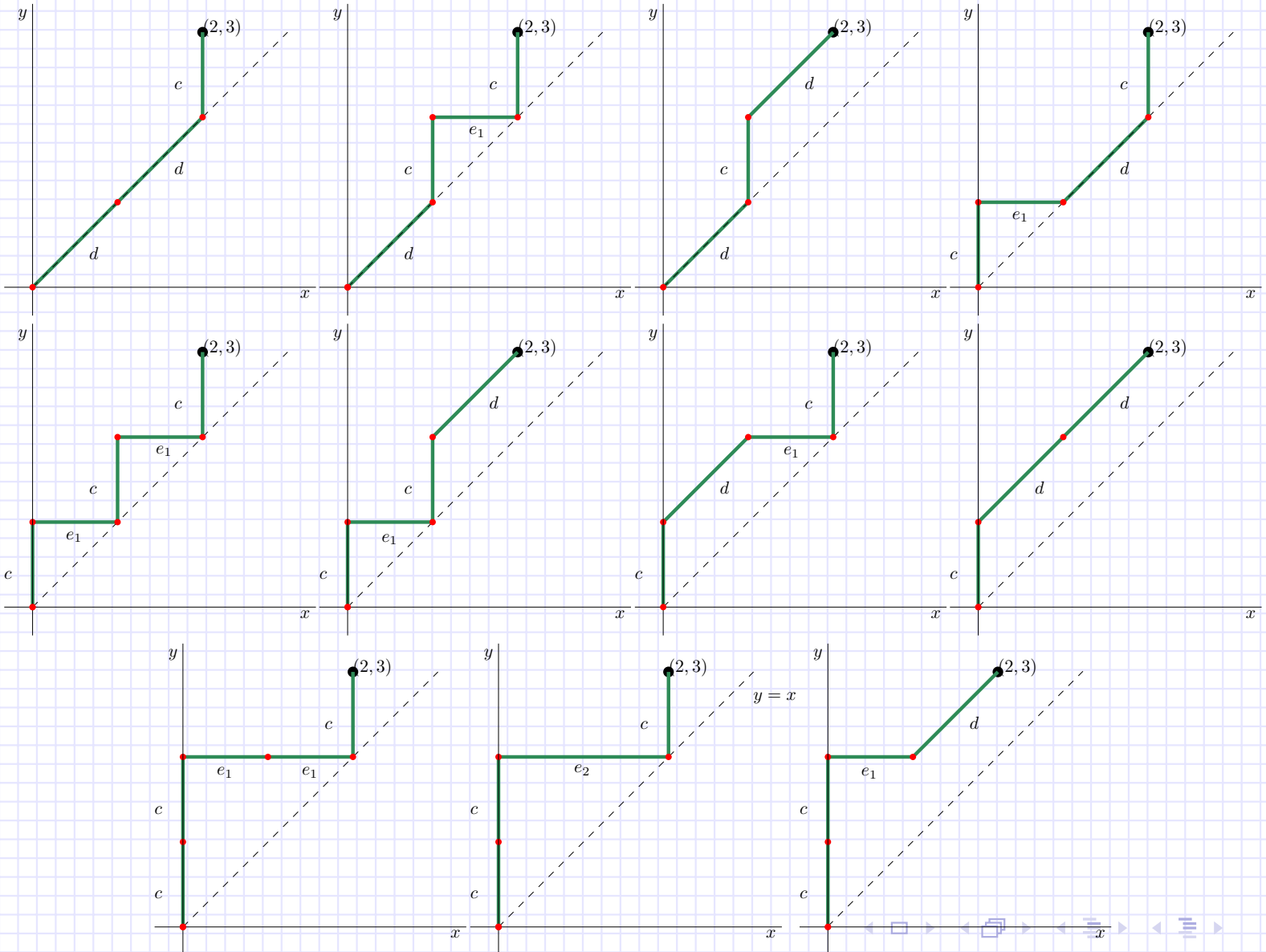
Example

$(0,0) \to (2,3)$



Schröder matrix

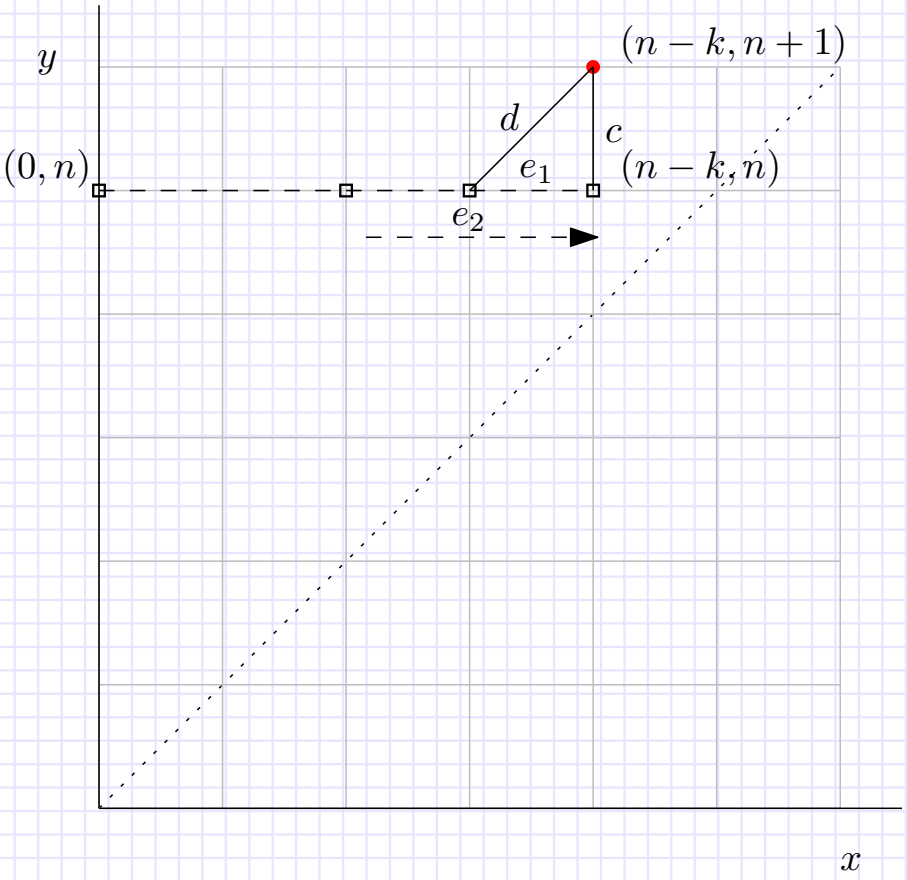
$$\mathcal{L}_4(3,1) = 2e_1^2c^3 + e_2c^3 + 5de_1c^2 + 3d^2c.$$



Schröder matrix

$$\mathcal{L}_4(n+1,k+1) = a_0\mathcal{L}_4(n,k) + a_1\mathcal{L}_4(n,k+1) + a_2\mathcal{L}_4(n+1,k+2) + \cdots + a_n\mathcal{L}_4(n,n), \quad n,k \geq 0;$$

where $a_0 = c, \quad a_1 = ce_1 + d, \quad a_2 = e_1(ce_1 + d) + e_2c, \ldots.$



Schröder matrix

In general,

$$a_n = e_1 a_{n-1} + e_2 a_{n-2}$$

for every integer $n \geq 2$.

$$A^{(4)}(x) = \sum_{i=0}^{\infty} a_i x^i = \frac{c + dx}{1 - e_1 x - e_2 x^2}.$$

If the point is on line $y = x$, we have

$$\mathcal{L}_4(n+1, 0) = b_0 \mathcal{L}_4(n, 0) + b_1 \mathcal{L}_4(n, 1) + b_2 \mathcal{L}_4(n+1, 2) + \cdots + b_n \mathcal{L}_4(n, n), \quad n, k \geq 0.$$

Note that $b_0 = d, \quad b_1 = de_1, \quad b_2 = d(e_1^2 + e_2), \ldots$. In general, $b_n = e_1 b_{n-1} + e_2 b_{n-2}$ for every integer $n \geq 2$. Therefore,

$$Z^{(4)}(x) = \sum_{i=0}^{\infty} b_i x^i = \frac{d}{1 - e_1 x - e_2 x^2}.$$

Schröder matrix

In general,

$$a_n = e_1 a_{n-1} + e_2 a_{n-2}$$

for every integer $n \geq 2$.

$$A^{(4)}(x) = \sum_{i=0}^{\infty} a_i x^i = \frac{c + dx}{1 - e_1 x - e_2 x^2}.$$

If the point is on line $y = x$, we have

$$\mathcal{L}_4(n+1, 0) = b_0 \mathcal{L}_4(n, 0) + b_1 \mathcal{L}_4(n, 1) + b_2 \mathcal{L}_4(n+1, 2) + \cdots + b_n \mathcal{L}_4(n, n), \quad n, k \geq 0.$$

Note that $b_0 = d, \quad b_1 = de_1, \quad b_2 = d(e_1^2 + e_2), \ldots$. In general, $b_n = e_1 b_{n-1} + e_2 b_{n-2}$ for every integer $n \geq 2$. Therefore,

$$Z^{(4)}(x) = \sum_{i=0}^{\infty} b_i x^i = \frac{d}{1 - e_1 x - e_2 x^2}.$$

Schröder matrix

In general,

$$a_n = e_1 a_{n-1} + e_2 a_{n-2}$$

for every integer $n \geq 2$.

$$A^{(4)}(x) = \sum_{i=0}^{\infty} a_i x^i = \frac{c + dx}{1 - e_1 x - e_2 x^2}.$$

If the point is on line $y = x$, we have

$$\mathcal{L}_4(n+1, 0) = b_0 \mathcal{L}_4(n, 0) + b_1 \mathcal{L}_4(n, 1) + b_2 \mathcal{L}_4(n+1, 2) + \cdots + b_n \mathcal{L}_4(n, n), \quad n, k \geq 0.$$

Note that $b_0 = d, \quad b_1 = de_1, \quad b_2 = d(e_1^2 + e_2), \ldots$. In general, $b_n = e_1 b_{n-1} + e_2 b_{n-2}$ for every integer $n \geq 2$. Therefore,

$$Z^{(4)}(x) = \sum_{i=0}^{\infty} b_i x^i = \frac{d}{1 - e_1 x - e_2 x^2}.$$

Schröder matrix

A-sequence and Z-sequence

Theorem

If $c \neq 0$ then the matrix $\mathcal{S}_4(c, d, e_1, e_2)$ is a Riordan array given by

$$\mathcal{S}_4(c, d, e_1, e_2) = \left(\frac{c}{c + dx}, x \frac{1 - e_1 x - e_2 x^2}{c + dx} \right)^{-1}.$$

Theorem

The inverse of the Riordan matrix $\mathcal{H}_4$ and the Riordan matrix $\mathcal{S}_4$ are related by

$$\mathcal{H}_4(1, b, c_1, c_2)^{-1} = \mathcal{S}_4(1, -b, -c_1, -c_2).$$

General Case

- $\mathbb{J}_{m+2}(n,k)$:= the set of weighted paths from the point $(0,0)$ to the point $(n-k,n)$, with step set

$$R_{m+2} = \{V = (0,1), D = (1,1), H_1 = (1,0), H_2 = (2,0), \ldots, H_m = (m,0)\},$$

  where each step is labelled with weights $c, d, e_1, \ldots, e_m$, respectively, and the path never falls below $y = x$ and its last step is not horizontal.
- $\mathcal{L}_{m+2}(n,k)$:= total sum of the weights of all weight paths in $\mathbb{J}_{m+2}(n,k)$.

Schröder matrix

Lemma

The numbers $\mathcal{L}_{m+2}(n,k)$ satisfy the following recurrence relation

$$\boxed{\begin{aligned}\mathcal{L}_{m+2}(n+1,k+1) = a_0\mathcal{L}_{m+2}(n,k) + a_1\mathcal{L}_{m+2}(n,k+1) \\ + \cdots + a_n\mathcal{L}_{m+2}(n,n), \quad n,k \geq 0.\end{aligned}}$$

where $a_n = e_1 a_{n-1} + e_2 a_{n-2} + \cdots + e_m a_{n-m}, \quad n \geq m,$

$$a_0 = c, \quad a_1 = ce_1 + d, \quad a_i = \sum_{j=0}^{i-1} e_{i-j} a_j, \quad 2 \leq i \leq m-1.$$

$$\mathcal{L}_{m+2}(n+1,0) = b_0\mathcal{L}_{m+2}(n,0) + b_1\mathcal{L}_{m+2}(n,1) + \cdots + b_n\mathcal{L}_{m+2}(n,n),$$

where $b_n = e_1 b_{n-1} + e_2 b_{n-2} + \cdots + e_m b_{n-m}$, *for every* $n \geq m$ *and*

$$b_0 = d, \quad b_i = \sum_{j=0}^{i-1} e_{i-j} b_j, \quad 1 \leq i \leq m-1.$$

Schröder matrix

Lemma

The generating functions $A^{(m+2)}(x)$ and $Z^{(m+2)}(x)$ of the sequences $(a_i)_{i\in\mathbb{N}}$ and $(b_i)_{i\in\mathbb{N}}$ are

$$A^{(m+2)}(x) := \sum_{i=0}^{\infty} a_i x^i = \frac{c + dx}{1 - e_1 x - e_2 x^2 - \cdots - e_m x^m},$$

and

$$Z^{(m+2)}(x) := \sum_{i=0}^{\infty} b_i x^i = \frac{d}{1 - e_1 x - e_2 x^2 - \cdots - e_m x^m}.$$

Schröder matrix

Theorem

The infinite matrix $\mathcal{S}_{m+2}$ is a Riordan matrix where $A^{(m+2)}(x)$ and $Z^{(m+2)}(x)$ are the generating functions of the A-sequence and Z-sequence, respectively. Moreover, if $c \neq 0$ then the matrix $\mathcal{S}_{m+2}(c, d, e_1, e_2, \ldots, e_m)$ is a Riordan array given by

$$\mathcal{S}_{m+2}(c, d, e_1, e_2, \ldots, e_m) = \left(\frac{c}{c+dx}, x\frac{1 - e_1 x - e_2 x^2 - \cdots - e_m x^m}{c+dx} \right)^{-1}.$$

Schröder matrix

Theorem

The inverse of the Riordan matrix $\mathcal{H}_{m+2}$ and the Riordan matrix $\mathcal{S}_{m+2}$ are related by

$$\mathcal{H}_{m+2}(1, b, c_1, c_2, \ldots, c_m)^{-1} = \mathcal{S}_{m+2}(1, -b, -c_1, -c_2, \ldots, -c_m).$$

J. L. R., V. Sirvent. Generalized Schröder matrix and its combinatorial interpretation. Linear and Multilinear Algebra, To Appear, 2017.

Outline

1. Lattice Path - Pascal Matrix - Delannoy Matrix ✓
2. $(a, b, c, d)-$weighted paths ✓
3. The General Case ✓
4. Schröder Matrix ✓
5. Some additional comments

Some additional comments

Given a positive real sequence $\mathcal{A} = \{a_n\}_{n \geq 0}$, we say that $\mathcal{A}$ is

- unimodal if there exists an integer $0 \leq j$ such that $a_0 \leq \cdots \leq a_{j-1} \leq a_j \geq a_{j+1} \geq \cdots$.
- The sequence $\mathcal{A}$ is called log-concave if it satisfies, for all $1 \leq j$, $a_j^2 \geq a_{j-1} a_{j+1}$.

Some additional comments

Example

$\left\{ \binom{n}{k} \right\}_{0 \le k \le n}$

$$
\begin{array}{ccccc}
\binom{0}{0} & & & & \\
\binom{1}{0} & \binom{1}{1} & & & \\
\binom{2}{0} & \binom{2}{1} & \binom{2}{2} & & \\
\binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} & \\
 & & \vdots & & \\
\binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{n}
\end{array}
$$

Some additional comments

Example

$$\left\{ \binom{n-k}{k} \right\}_{0 \le k \le \lfloor n/2 \rfloor}$$

$$\begin{array}{ccccc}
\binom{0}{0} & & & & \\
\binom{1}{0} & \binom{1}{1} & & & \\
\binom{2}{0} & \binom{2}{1} & \binom{2}{2} & & \\
\binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} & \\
 & & \vdots & & \\
\binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{n}
\end{array}$$

S. Tanny, M. Zuker. On a unimodality sequence of binomial coefficients. Discrete Math. 9(1974), 79–89.

Some additional comments

Example

Belbachir and Szalay (2008): any sequence of binomial coefficients lying along a ray in Pascal's triangle is log-concave, thus unimodal $\left\{\binom{u_0+\alpha k}{v_0+\beta k}\right\}_k$

$$
\begin{array}{ccccc}
\binom{0}{0} & & & & \\
\binom{1}{0} & \binom{1}{1} & & & \\
\binom{2}{0} & \binom{2}{1} & \binom{2}{2} & & \\
\binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} & \\
 & & \vdots & & \\
\binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{n}
\end{array}
$$

H. Belbachir, L. Szalay. Unimodal rays in the ordinary and generalized Pascal triangles. J. Integer Seq. 11(2008), Article. 08.2.4.

X.-T. Su, Y. Wang. On unimodality problems in Pascal's triangle. Electron. J. Combin. 15(2008), # R113.

Y. Yu. Confirming two conjectures of Su and Wang on binomial coefficients. Adv. in Appl. Math. 43(2009), 317–322.

Some additional comments

Example

Stirling Matrix $\left\{\left\{ {n \atop m} \right\}\right\}_{n,k\geq 0}$

Log-concave: $\left\{\left\{ {n_0-ai \atop k_0+bi} \right\}\right\}_i$

B.-X. Zhu. Some positivities in certain triangular arrays. Proceedings of the American Mathematical Society. 142(9)(2014) 2943–2952.

Bonus Example!

Example

Adiprasito, Huh and Katz solved the Rota conjecture on the log-concavity of the characteristic polynomial of matroids.

Some additional comments

Example

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Bonus Example!

Example

Adiprasito, Huh and Katz solved the Rota conjecture on the log-concavity of the characteristic polynomial of matroids.

Some additional comments

Delannoy Matrix?

$$\mathcal{H}_2 = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 3 & 1 & & & \\ 1 & 5 & 5 & 1 & & \\ 1 & 7 & 13 & 7 & 1 & \\ 1 & 9 & 25 & 25 & 9 & 1 \\ & & \vdots & & & \end{bmatrix}$$

Conjecture

Let n, p, r be positive integers and $q \in \mathbb{Z}$, with $n \geq p$, $0 \leq p < r$ and $q + r > 0$ then the sequence $\{d(n - qk, p + rk)\}_{0 \leq k}$ satisfies:

- *If $q > 0$ then the sequence is log-concave and therefore unimodal.*
- *If $q < 0$ and $q < r$ then the sequence is log-concave and therefore unimodal.*
- *If $q < 0$ and $q = r$ then the sequence is log-concave and increasing.*
- *If $q < 0$ and $r < q$ then the sequence is increasing and asymptotically log-convex.*

Some additional comments

Theorem

Let $q \in \mathbb{Z}$, $r \in \mathbb{N}$, $p \in \mathbb{Z}^+$ with $0 \leq p < r$ and $q + r > 0$. Any sequence $\{d(n - qk, p + rk)\}_{0 \leq k \leq \lfloor (n-p)/(q+r) \rfloor}$ lying along all finite rays in the generalized Delannoy matrix is log-concave, thus unimodal.

(a,b,c,d)-Matrix? General case?

Some additional comments

Theorem

Let $q \in \mathbb{Z}$, $r \in \mathbb{N}$, $p \in \mathbb{Z}^+$ with $0 \leq p < r$ and $q + r > 0$. Any sequence $\{d(n - qk, p + rk)\}_{0 \leq k \leq \lfloor (n-p)/(q+r) \rfloor}$ lying along all finite rays in the generalized Delannoy matrix is log-concave, thus unimodal.

(a,b,c,d)-Matrix? General case?

Some additional comments

Delannoy Matrix

$$\mathcal{H}_2 = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 3 & 1 & & & \\ 1 & 5 & 5 & 1 & & \\ 1 & 7 & 13 & 7 & 1 & \\ 1 & 9 & 25 & 25 & 9 & 1 \\ & & \vdots & & & \end{bmatrix}$$

- Central Delannoy Numbers:
  $\{D_n\}_n = \{1, 3, 13, 63, 321, 1683, 8989, 48639, \dots\}$
- Liu and Wang, 2007: proved that the central Delannoy sequence is log-convex.
- Given a positive real sequence $\mathcal{A} = \{a_n\}_{n \geq 0}$, we say that $\mathcal{A}$ is log-convex if it satisfies, for all $1 \leq j$, $a_j^2 \leq a_{j-1} a_{j+1}$.

Some additional comments

Inverse of Delannoy matrix (Schröder matrix)

$$\mathcal{H}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \\ -1 & 1 & 0 & 0 & 0 & 0 & \\ 2 & -3 & 1 & 0 & 0 & 0 & \\ -6 & 10 & -5 & 1 & 0 & 0 & \\ 22 & -38 & 22 & -7 & 1 & 0 & \\ -90 & 158 & -98 & 38 & -9 & 1 & \cdots \\ \vdots & & \vdots & & \vdots & & \ddots \end{bmatrix}.$$

- Central sequence:

$$\{F_n\}_n = \{1, 3, 22, 194, 1838, 18082, 182054, 1861890, \dots\}$$

- F_n counts the number of lattice paths from the point $(0,0)$ to the point $(n, 2n)$, whose step set is $\{V = (0,1), D = (1,1), H = (1,0)\}$ such that its last step is not a horizontal step and that never falling below line $y = x$.

Some additional comments

Inverse of Delannoy matrix (Schröder matrix)

$$\mathcal{H}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \\ -1 & 1 & 0 & 0 & 0 & 0 & \\ 2 & -3 & 1 & 0 & 0 & 0 & \\ -6 & 10 & -5 & 1 & 0 & 0 & \\ 22 & -38 & 22 & -7 & 1 & 0 & \\ -90 & 158 & -98 & 38 & -9 & 1 & \cdots \\ \vdots & & \vdots & & \vdots & & \ddots \end{bmatrix}.$$

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Some additional comments

$F_2 = 22$

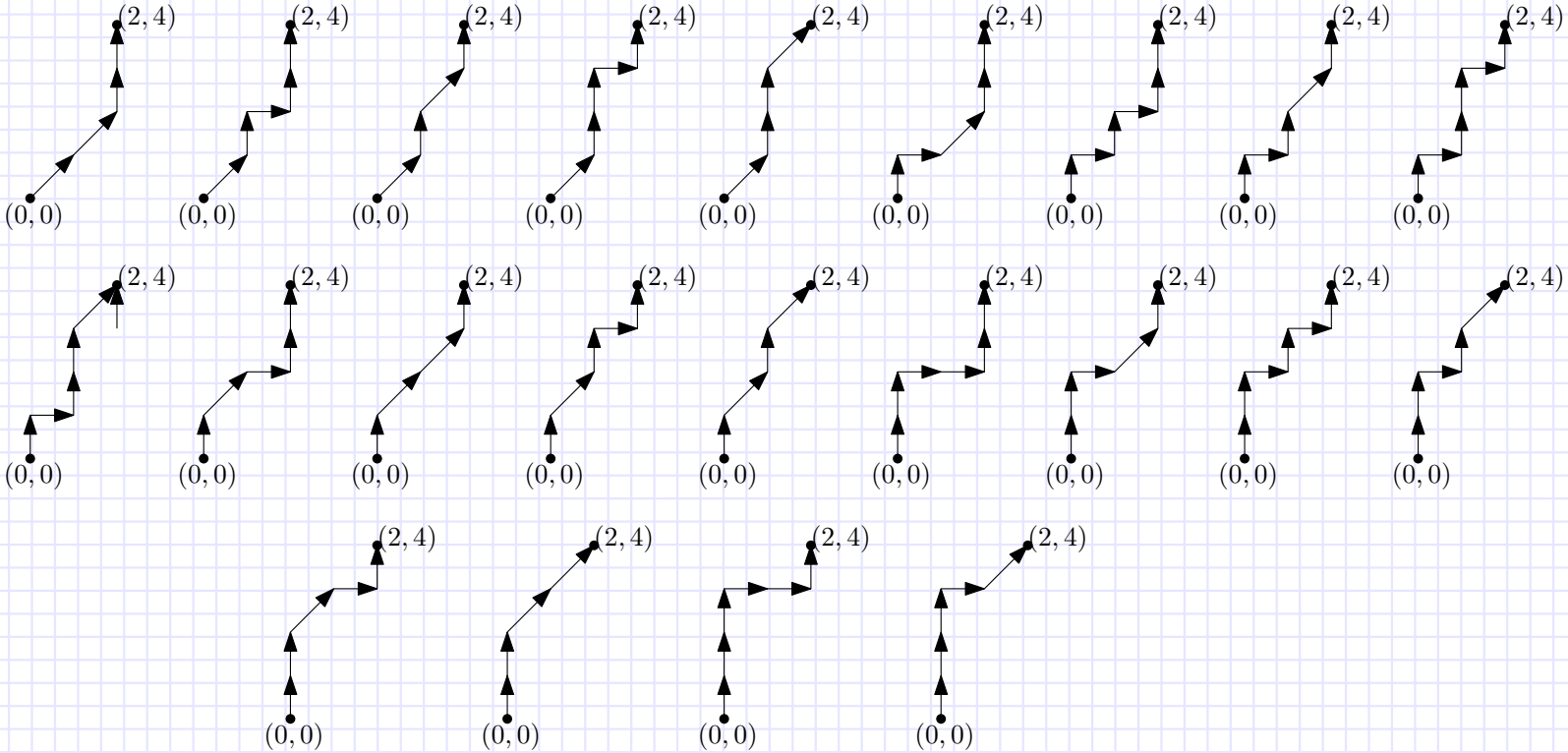

Figure: Combinatorial interpretation for F_n.

Some additional comments

Theorem

The sequence F_n is log-convex.

Proof.

Computational Method...

$$F_n = \sum_{j=0}^{n} \left(\frac{n}{2n-j} \binom{2n-1}{j} \binom{3n-j-1}{2n-1} + \frac{n+1}{2n-j} \binom{2n-1}{j} \binom{3n-j-2}{2n-1} \right), \quad n \geq 0.$$

□

Some additional comments

Theorem

The sequence F_n is log-convex.

Proof.

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$\square$

Some additional comments

From the Kauers's multivariate guessing library we obtain that

$$p_n F_n = q_n F_{n-1} + r_n F_{n-2}, \quad n \geq 2$$

with the initial values $F_0 = 1$ and $F_1 = 3$, and

$$\begin{aligned} p_n &= 20n^4 - 72n^3 + 85n^2 - 39n + 6, \\ q_n &= 220n^4 - 902n^3 + 1272n^2 - 711n + 129, \\ r_n &= 20n^4 - 92n^3 + 89n^2 + 36n - 45. \end{aligned}$$

Some additional comments

Let $\{z_n\}_n$ be a sequence such that

$$a(n)z_{n+1} = b(n)z_n + c(n)z_{n-1}, \tag{1}$$

where $a(n), b(n)$ and $c(n)$ are positive for $n \geq 1$.

Lemma (Liu and Wang, 2007)

Let $\{z_n\}_n$ be defined by (1) and

$$\lambda_n = \frac{b(n) + \sqrt{b^2(n) + 4a(n)c(n)}}{2a(n)}.$$

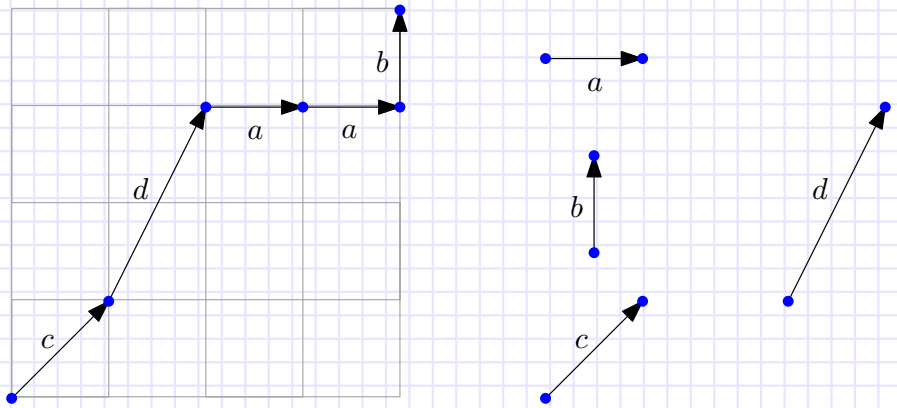
Suppose that z_0, z_1, z_2, z_3 is log-convex and that the inequality

$$a(n)\lambda_{n-1}\lambda_{n+1} - b(n)\lambda_{n-1} - c(n) \geq 0$$

is true for $n \geq 2$. Then the sequence $\{z_n\}_n$ is log-convex.

Wolfram Mathematica

$(1,1,1,1)$-weighted paths



$$\mathcal{H}_3(1,1,1,1) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 1 & 0 & 0 & 0 & 0 & \\ 1 & 3 & 1 & 0 & 0 & 0 & \\ 1 & 6 & 5 & 1 & 0 & 0 & \\ 1 & 9 & 15 & 7 & 1 & 0 & \\ 1 & 12 & 33 & 28 & 9 & 1 & \\ \vdots & & \vdots & & & & \ddots \end{bmatrix}$$

Central sequence: $\left\{ d^{(3)}_{2n,n} \right\}_{n\in\mathbb{N}} = \{1, 3, 15, 81, 459, 2673, \dots\}$

$(1,1,1,1)$-weighted paths

Theorem (Barry, 2013)

Let $(d(t), h(t)) = (d(t), tf(t))$ be an element of the Riordan group of matrices $\mathcal{R}$ (with $f(0) \neq 0$). Let $d_{n,k}$ denote the (n,k)-th element of this matrix, and let $v(t)$ denote the power series (with $v(0) = 0$)

$$v(t) = \overline{\left(\frac{t}{f(t)}\right)}.$$

Then the generating function of the central term sequence $d_{2n,n}$ is given by

$$\frac{d(v(t))}{f(v(t))}\frac{d}{dt}v(t).$$

The generating function of the central sequence $\left\{d^{(3)}_{2n,n}\right\}_{n\in\mathbb{N}} = \{1, 3, 15, 81, 459, 2673, \ldots\}$ is

$$\sum_{n=0}^{\infty} d^{(3)}_{2n,n} z^n = \sum_{n=0}^{\infty} \mathcal{D}_3(n,n) z^n = \frac{1}{\sqrt{1-6z-3z^2}}.$$

$(1,1,1,1)$-weighted paths

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Let $(d(t), h(t)) = (d(t), tf(t))$ be an element of the Riordan group of matrices $\mathcal{R}$ (with $f(0) \neq 0$). Let $d_{n,k}$ denote the (n,k)-th element of this matrix, and let $v(t)$ denote the power series (with $v(0) = 0$)

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$(1, 1, 1, 1)$-weighted paths

- *Grand Motzkin path*:

$(1,1,1,1)$-weighted paths

$g_n^{(3,3)} :=$ the number of grand Motzkin path such that each horizontal step is colored with one of 3 colors and each up diagonal step is colored with one of 3 specific colors.

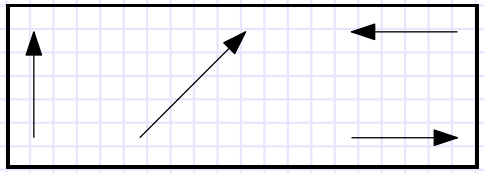
Corollary

The number of $3^H 3^U$-Grand Motzkin path is equal to the number of $(1,1,1,1)$-weighted paths from the point $(0,0)$ to the point (n,n), i.e.,

$$g_n^{(3,3)} = \mathcal{D}_3(n,n).$$

$(1,1,1,1)$-weighted paths

Dziemiańczuk, 2013:
Let $S(n,k)$ be the number of lattice paths from $(0,0)$ to (n,k) in the plane $\mathbb{Z}\times\mathbb{N}$ with set of steps



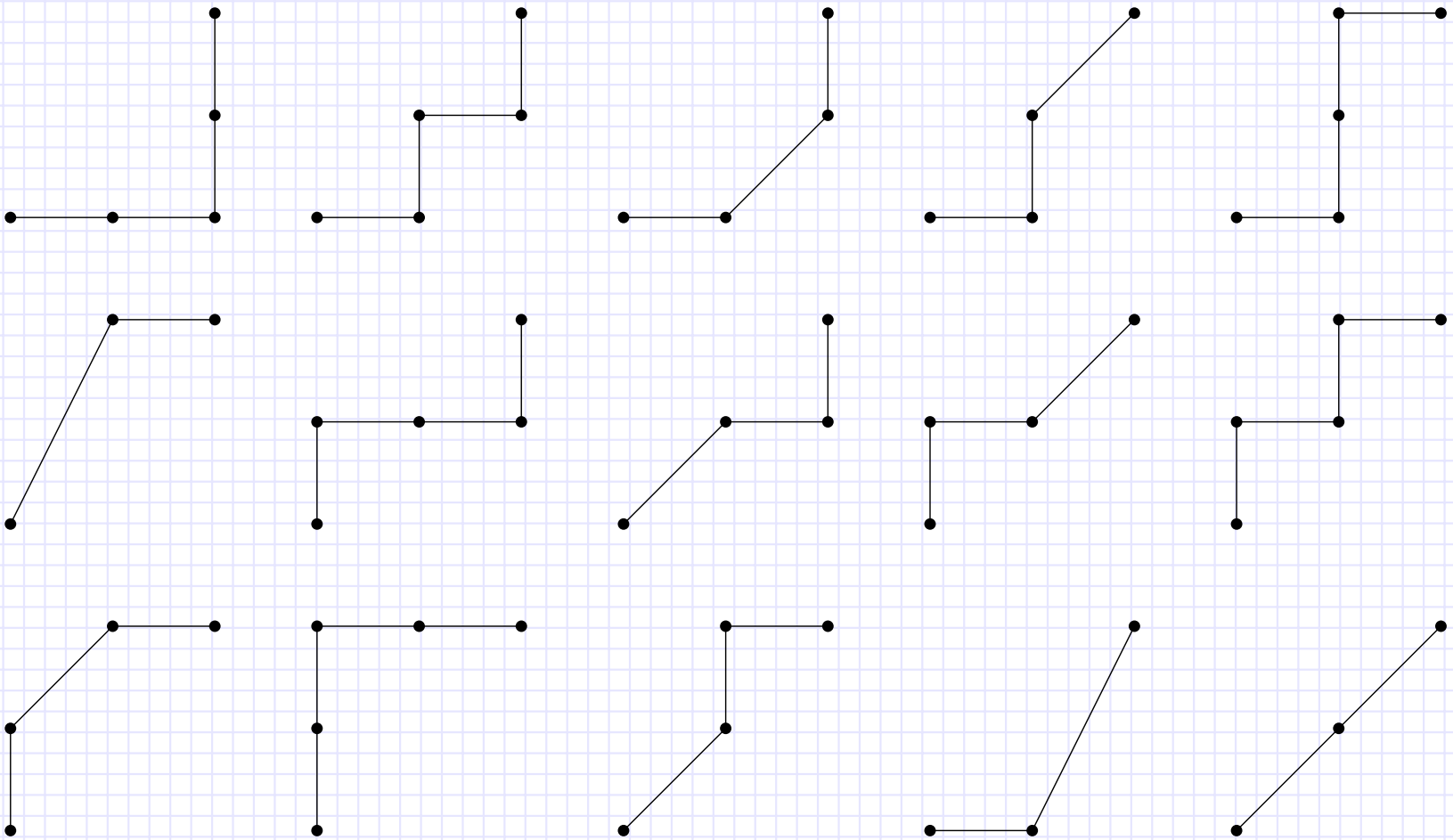
$$S = \{H=(1,0), V=(0,1), D_1=(1,1), D_2=(-1,1)\}$$

$$\sum_{n=0}^{\infty} S(0,n)z^n = \frac{1}{\sqrt{1-6z-3z^2}}.$$

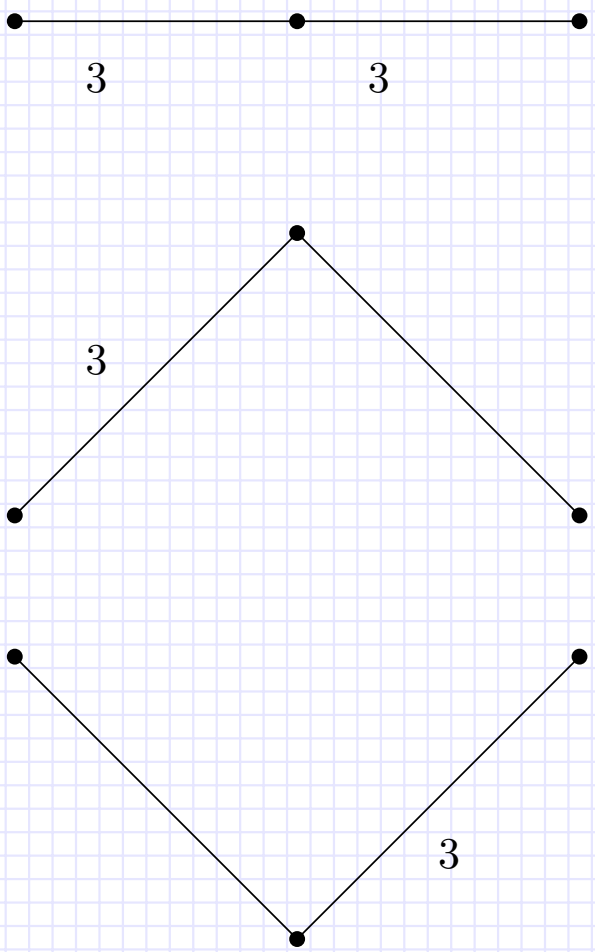
Therefore

$$\boxed{S(0,n) = g_n^{(3,3)} = \mathcal{D}_3(n,n).}$$

$(1, 1, 1, 1)$-weighted paths



$(1, 1, 1, 1)$-weighted paths



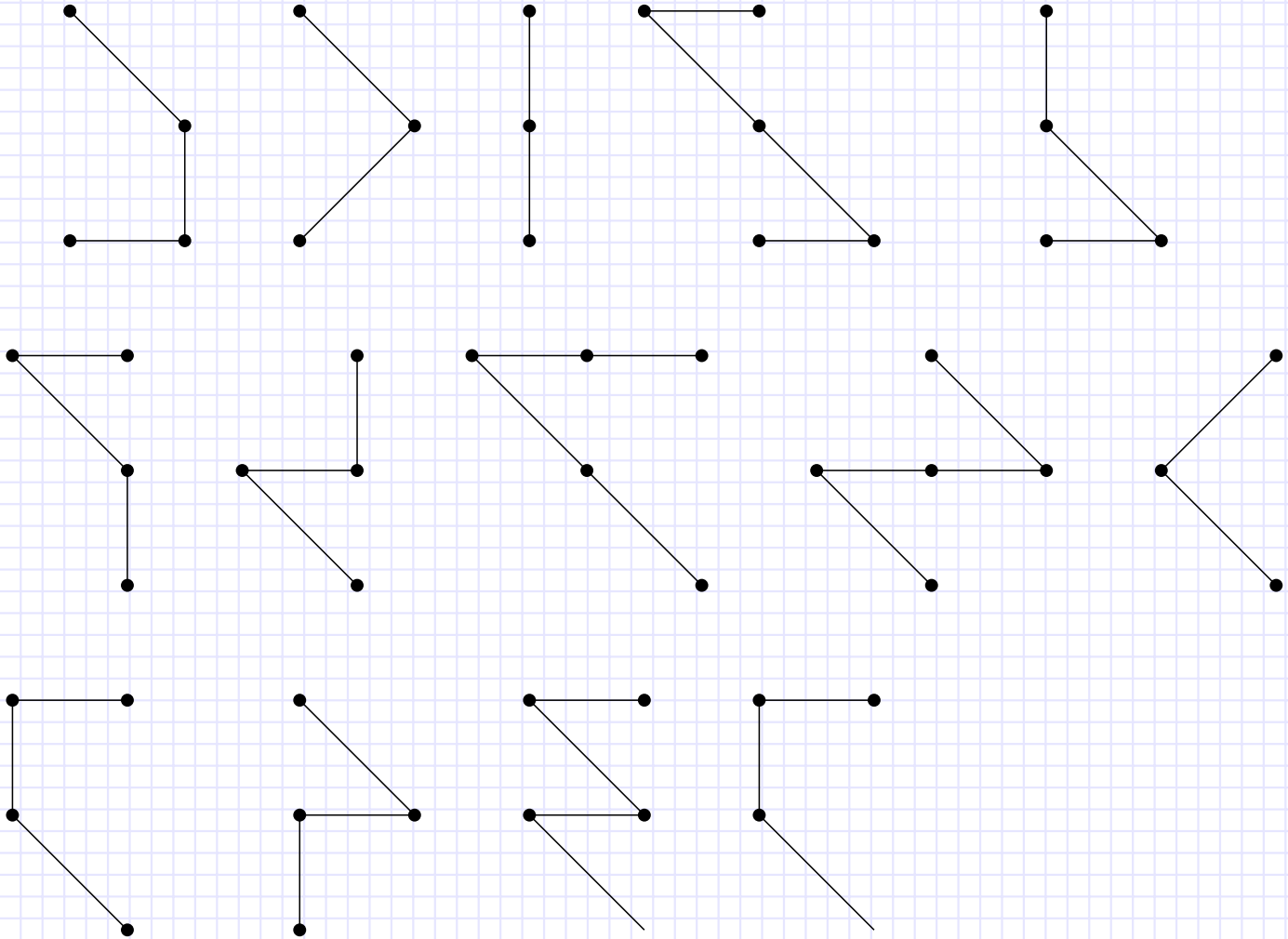
9

3

3

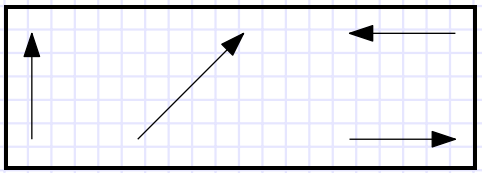
15

$(1, 1, 1, 1)$-weighted paths



Some additional comments

- Is there any Bijection?
- M. Dziemiańczuk. Counting lattice paths with four types of steps. Graphs Combin. (2013).



$$S = \{H = (1,0), V = (0,1), D_1 = (1,1), D_2 = (-1,1)\}$$

$$R(n,k) = \left[z^k\right] \frac{2^{n+2}(1+z)^n\sqrt{1-6z-3z^2}}{(1-z+\sqrt{1-6z-3z^2})^{n+2} - (1-z-\sqrt{1-6z-3z^2})^{n+2}},$$

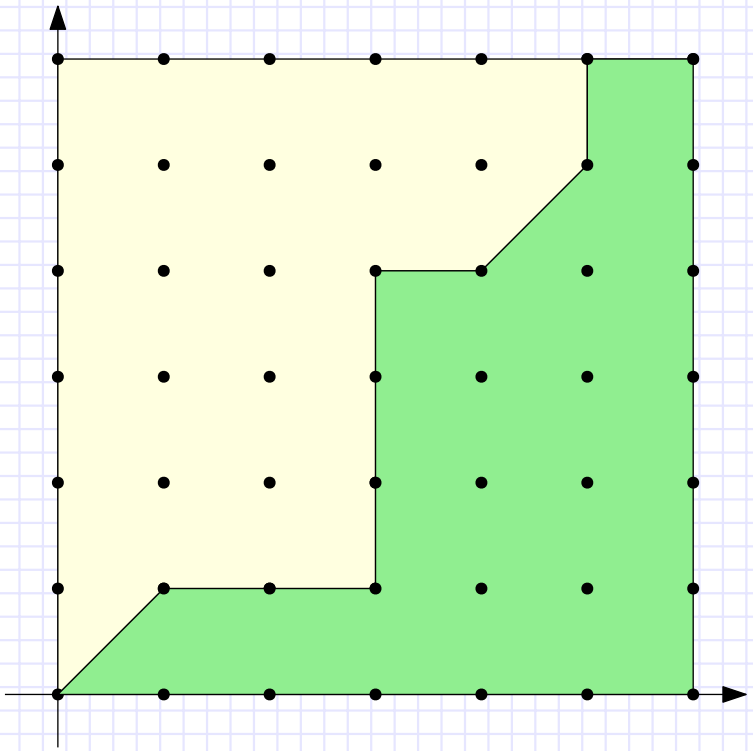
where $R(n,k)$ is the family of lattice paths from the origin to (n,k) whose points lie entirely in the integer rectangle of lattice points $\{(i,j) : 0 \leqslant i \leqslant n, 0 \leqslant j \leqslant k\}$.

Some additional comments

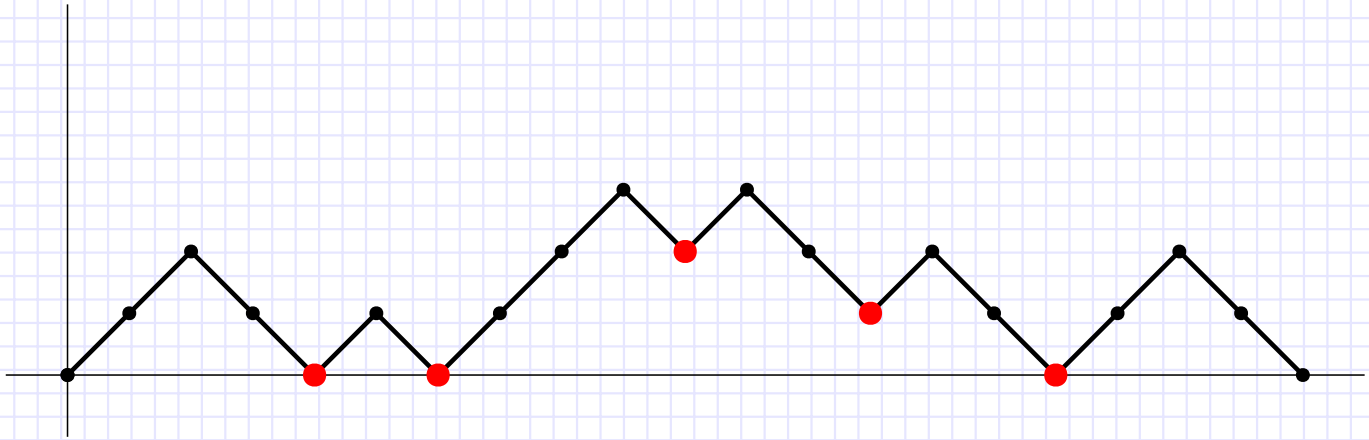
- Carlitz, L. **Some q-analogues of certain combinatorial numbers**, SIAM J. Math. Anal. 4(3), (1973)

$$A(n,r) = A(n-1,r-1) + q^n A(n,r-1) + q^r A(n-1,r)$$

- C. Wagner. **The Carlitz lattice path polynomials**, Discrete Mathematics 222 (2000) 291–298.



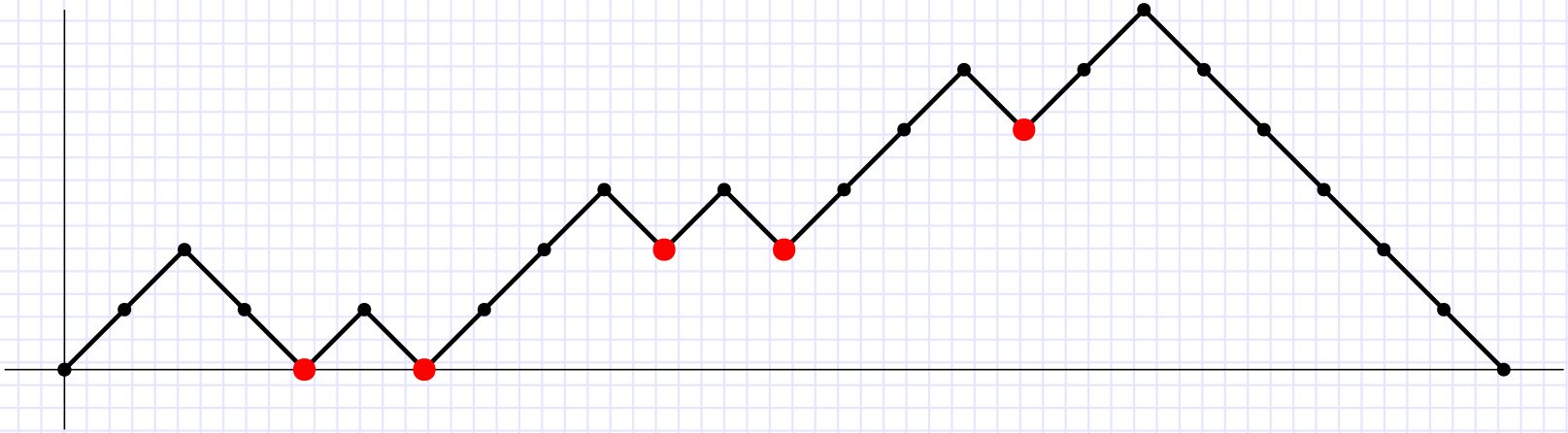
Some additional comments



The valleys and peaks of a Dyck path P are the local minima and local maxima, respectively. We say that a Dyck path P is non-decreasing if the y-coordinates of valleys of the path P form a non-decreasing sequence.

Some additional comments

Non-decreasing sequence Dyck path.

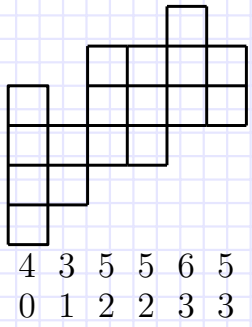


The number of non-decreasing Dyck paths of length $2n$ is the Fibonacci number F_{2n-1}.

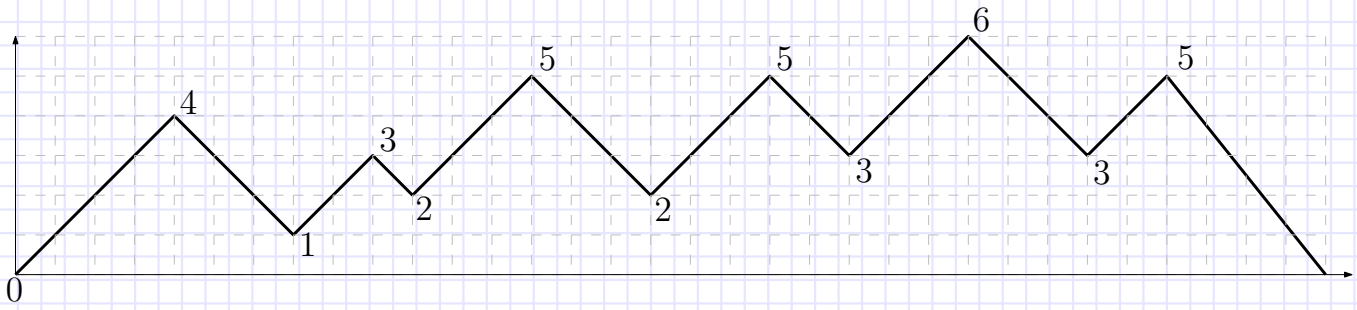
E. Barcucci, A. Del Lungo, S. Fezzi, and R. Pinzani, Nondecreasing Dyck paths and q-Fibonacci numbers. *Discrete Math.* 170(1–3)(1997), 211–217.

Some additional comments

Deutsch and Prodinger, 2003, give a bijection between the non-decreasing Dyck path of length $2n$ and the direct column-convex polyomino (dccp) of area n.



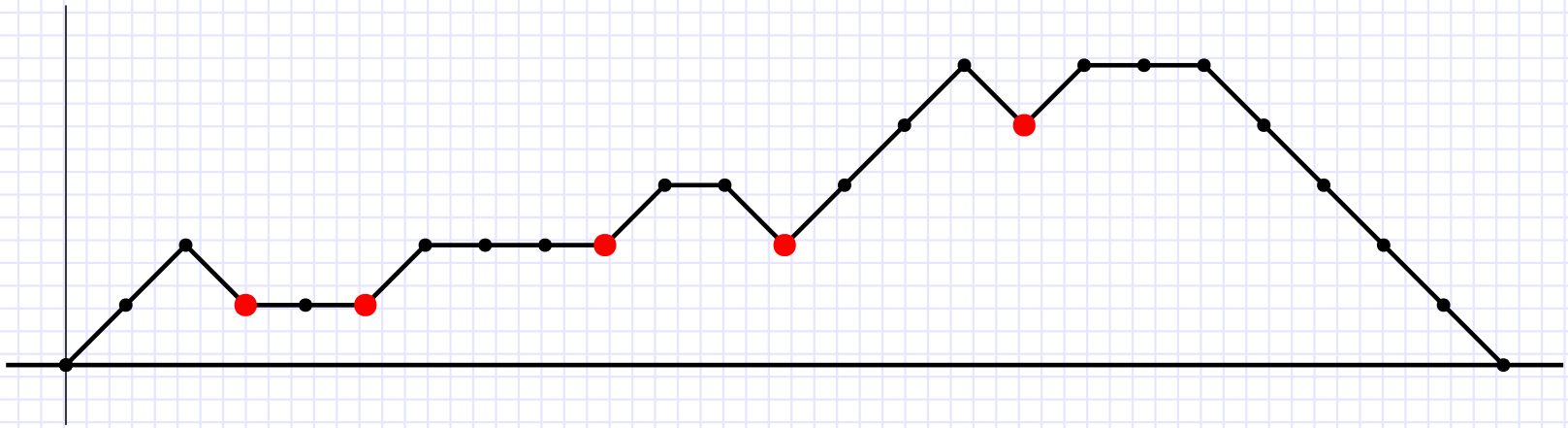
(a)



(b)

Some additional comments

The valleys of a Motzkin path P are the local minima. We say that a Motzkin path P is non-decreasing if the y-coordinates of all valleys (including also the left valleys and the right valleys) of the path P form a non-decreasing sequence.



Some additional comments

- $l(P) :=$ the length of P,
- $h(P) :=$ the number of horizontal steps of P
- $r(P) :=$ the number of rises (up-diagonal steps) of P
- $p(P) :=$ the number of peaks of P.

$$F(x,y,z,q) := \sum_{P \in \mathcal{NM}} x^{l(P)} y^{h(P)} z^{r(P)} q^{p(P)}.$$

Theorem

The generating function $F(x,y,z,q)$ is given by the following equation

$$F(x,y,z,q) = \frac{x(xy-1)(xzq+y)(x^2z-1)}{1-2xy+x^2y^2-2x^2z-qx^2z+2x^3yz+qx^3yz-x^4y^2z+x^4z^2-x^5yz^2}.$$

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Some additional comments

Proof.

Symbolic Method:

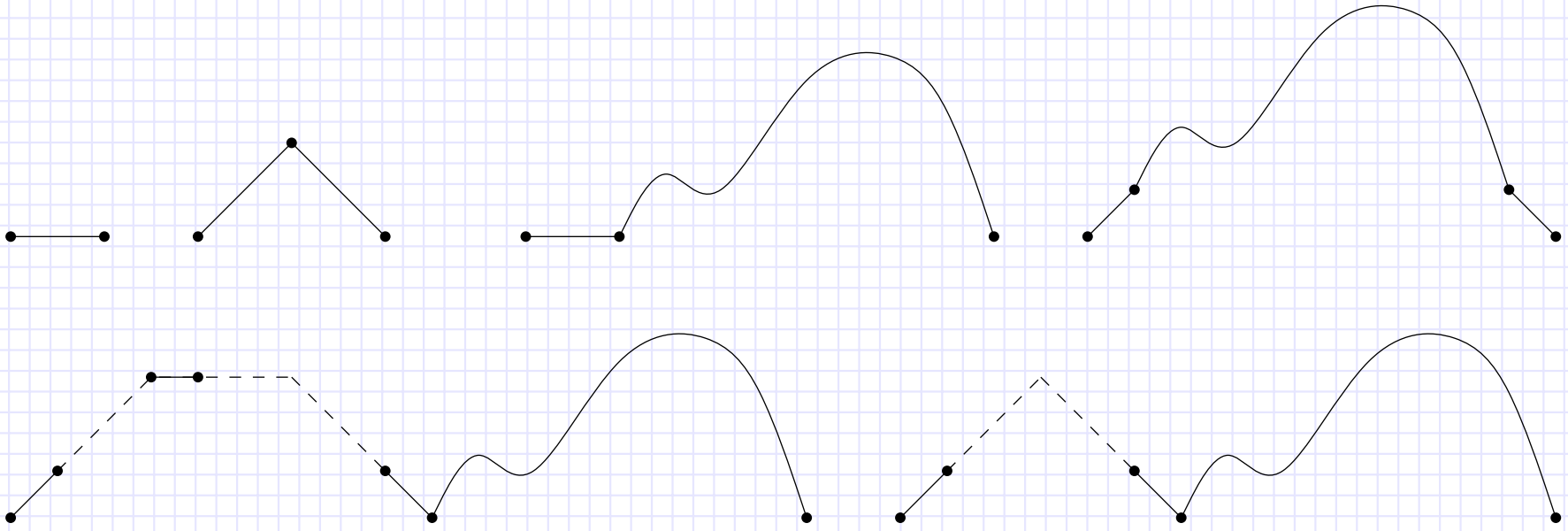

Figure: Factorizations of any non-decreasing Motzkin path.

□

Some additional comments

If $y = 1 = z = q$ we obtain the generating function for the non-decreasing Motzkin paths respect to the length:

$$\begin{aligned} F(x,1,1,1) &= \frac{x(x^2-1)^2}{1-2x-2x^2+3x^3-x^5} \\ &= x + 2x^2 + 4x^3 + 9x^4 + 21x^5 + 49x^6 + 115x^7 + \cdots \end{aligned}$$

If $y = 0$ and $q = 1 = z$ then

$$F(x,0,1,1) = \frac{x^2(1-x^2)}{1-3x^3+x^4} = \sum_{n=1}^{\infty} F_{2n-1} x^{2n}.$$

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Some additional comments

- A path which is a prefix of a non-decreasing Motzkin path is called non-decreasing Motzkin prefix.
- The prefixes of the classical Dyck paths are called Ballot paths
- $m_{n,k} :=$ number of non-decreasing Motzkin prefixes of length n and height k.
- $\mathbb{M} = [m_{n,k}]_{n,k \geq 0}$.

$$\mathbb{M} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 4 & 5 & 3 & 1 & 0 & 0 & 0 & 0 \\ 9 & 12 & 9 & 4 & 1 & 0 & 0 & 0 \\ 21 & 30 & 25 & 14 & 5 & 1 & 0 & 0 \\ 49 & 74 & 69 & 44 & 20 & 6 & 1 & 0 \\ 115 & 182 & 185 & 133 & 70 & 27 & 7 & 1 \\ \vdots & & & \vdots & & & & \vdots \end{pmatrix}$$

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Some additional comments

Example

$m_{4,2} = 9$ paths.

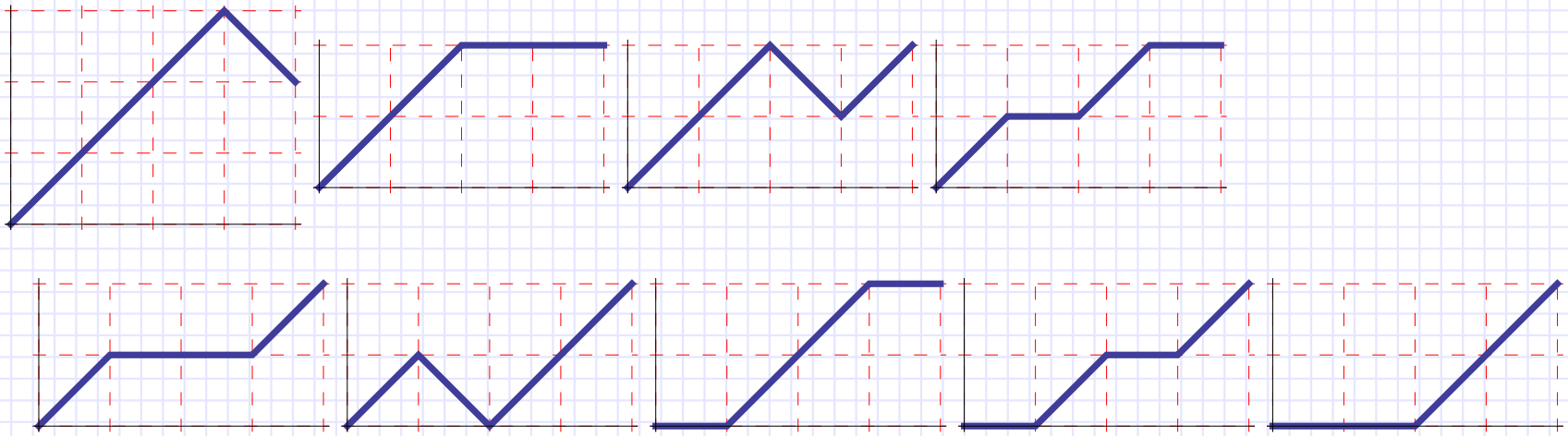

Figure: Non-decreasing Motzkin prefixes of length 4 and height 2.

Some additional comments

Theorem

Let $T_k(x)$ be the generating function defined be

$$T_k(x) := \sum_{n=0}^{\infty} d_{n,k} x^n.$$

Then

$$T_k(x) = \frac{1 - x - 2x^2 + x^3}{1 - 2x - 2x^2 + 3x^3 - x^5} \left(x \frac{(1-x)^2(1+x)}{1 - 2x - x^2 + 2x^3 - x^4} \right)^k.$$

Some additional comments

Theorem

The matrix $\mathbb{M}$ is a Riordan array given by

$$\mathbb{M} = \left(\frac{1 - x - 2x^2 + x^3}{1 - 2x - 2x^2 + 3x^3 - x^5}, x\frac{(1-x)^2(1+x)}{1 - 2x - x^2 + 2x^3 - x^4} \right).$$

Additional Properties of the non-decreasing Motzkin paths....

Some additional comments

Theorem

The matrix $\mathbb{M}$ is a Riordan array given by

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Additional Properties of the non-decreasing Motzkin paths....

Thank you!!!
Gracias!!

Figure: Churros and Chocolate; Chocolatería San Ginés, 17-Jul-2017!