Stieltjes and Hamburger moment sequences in combinatorics

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17-20 July 2017

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Outline

Background

- The Problem of Moments
- continued fractions

Thomas Joannes Stieltjes
(1856-1894)

- Stieltjes, Recherches sur les fractions continues, Ann. Fac. Sci. Toulouse Sci. Math. Sci. Phys., 1894.
- Widder, The Laplace Transform, 1941.
- Shohat and Tamarkin, The Problem of Moments, 1943.
- Akhiezer, The classical moment problem and some related questions in analysis, 1964.

Stieltjes and Hamburger moment sequence

Definition

Let $\alpha = (a_n)_{n\geq 0}$ be an infinite sequence of real numbers.
Call α a Stieltjes (Hamburger, resp.) moment (SM (HM) for short) sequence if it has the form

$$a_n = \int_0^{+\infty} x^n d\mu(x),$$

where μ is a positive Borel measure on $x \geq 0$ ($x \in \mathbb{R}$, resp.).

Example

The factorial numbers $n!$ form a SM,

$$n! = \int_0^{\infty} x^n d(1 - e^{-x}).$$

Classical results of Stieltjes moment sequence

Let $\alpha = (a_n)_{n\geq 0}$ be real numbers. Define the Hankel matrix $H(\alpha)$ of α by

$$H(\alpha) := [a_{i+j}]_{i,j\geq 0} = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Call a matrix totally positive if all its minors are nonnegative.

Theorem

$$\begin{aligned} & \alpha \text{ is SM} \\ \iff\ & H(\alpha) \text{ and } H(\overline{\alpha}) \text{ are positive semi-definite} \\ \iff\ & H(\alpha) \text{ is totally positive.} \end{aligned}$$

Example

Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ form a SM.

$$\det[C_{i+j}]_{0\leq i,j\leq n} = \det[C_{i+j+1}]_{0\leq i,j\leq n} = 1.$$

Classical results of Hamburger moment sequence

Theorem

Let $\alpha = (a_n)_{n\geq 0}$ be an infinite sequence of real numbers.

$$\begin{aligned} & \alpha \text{ is HM} \\ \Longleftrightarrow\ & H(\alpha) \text{ is positive semi-definite} \\ \Longleftrightarrow\ & \textstyle\sum_{i,j\geq 0} a_{i+j} x_i x_j \geq 0. \end{aligned}$$

- $\alpha = (a_n)_{n\geq 0}$ is SM

$\Longleftrightarrow$ Both $\alpha = (a_n)_{n\geq 0}$ and $\overline{\alpha} = (a_{n+1})_{n\geq 0}$ are HM.

- SM $\Rightarrow$ HM.

Shohat and Tamarkin, The Problem of Moments, Amer. Math. Soc., New York, 1943.

Classical results of Hamburger moment sequence

Theorem

Let $\alpha = (a_n)_{n\geq 0}$ be an infinite sequence of real numbers. Then α is a HM sequence if all Hankel determinants

$$h_n(\alpha) := \det[a_{i+j}]_{0\leq i,j\leq n}$$

are nonnegative and $h_N = 0$ implies $h_n = 0$ for $n \geq N$.

Example

The Fibonacci sequence $\mathfrak{F} = (F_n)_{n\geq 0}$ is HM.

$$F_0 = F_1 = 1 \text{ and } F_n = F_{n-1} + F_{n-2},$$

Shohat and Tamarkin, The Problem of Moments, Amer. Math. Soc., New York, 1943.

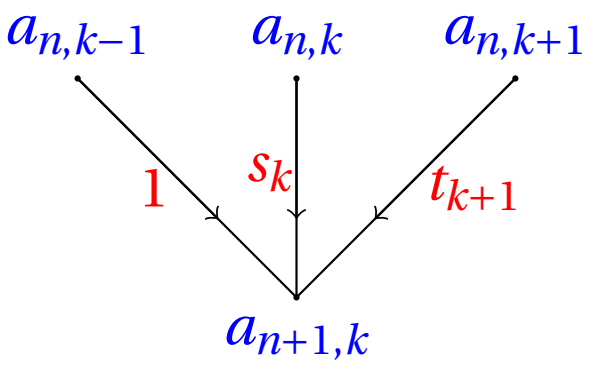
Catalan-like numbers

Definition

Let $\mathfrak{s} = (s_k)_{k\geq 0}$ and $\mathfrak{t} = (t_{k+1})_{k\geq 0}$ be nonnegative sequences.
Define the recursive matrix $A^{(\mathfrak{s},\mathfrak{t})} = [a_{n,k}]_{n,k\geq 0}$ by

$$a_{0,0} = 1, \quad a_{n+1,k} = a_{n,k-1} + s_k a_{n,k} + t_{k+1} a_{n,k+1}, \tag{1}$$

where $a_{n,k} = 0$ unless $n \geq k \geq 0$.
Call $a_n := a_{n,0}$ the Catalan-like numbers.



Example of recursive matrix

$$A^{(\mathfrak{s},\mathfrak{t})} = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 2 & 3 & 1 & & & \\ 5 & 9 & 5 & 1 & & \\ 14 & 28 & 20 & 7 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where $\mathfrak{s} = (1,2,2,\ldots)$, $\mathfrak{t} = (1,1,1,\ldots)$.

- $C_n = (1,1,2,5,14,\ldots)$.

Example of recursive matrix

$$A^{(\mathfrak{s},\mathfrak{t})} = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 2 & 2 & 1 & & & \\ 4 & 5 & 3 & 1 & & \\ 9 & 12 & 9 & 4 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where $\mathfrak{s} = \mathfrak{t} = (1, 1, 1, \ldots)$.

- $M_n = (1, 1, 2, 4, 9, \ldots)$.

Rewrite the recursive relation (1) as

$$\begin{bmatrix} a_{1,0} & a_{1,1} & & & \\ a_{2,0} & a_{2,1} & a_{2,2} & & \\ a_{3,0} & a_{3,1} & a_{3,2} & a_{3,3} & \\ & & \cdots & & \ddots \end{bmatrix} = \begin{bmatrix} a_{0,0} & & & \\ a_{1,0} & a_{1,1} & & \\ a_{2,0} & a_{2,1} & a_{2,2} & \\ & \cdots & & \ddots \end{bmatrix} \begin{bmatrix} s_0 & 1 & & \\ t_1 & s_1 & 1 & \\ & t_2 & s_2 & \ddots \\ & & \ddots & \ddots \end{bmatrix},$$

or briefly,

$$\overline{A} = AJ$$

where $\overline{A}$ is obtained from A by deleting the 0th row and J is the tridiagonal matrix

$$J := J^{\sigma,\tau} = \begin{bmatrix} s_0 & 1 & & & \\ t_1 & s_1 & 1 & & \\ & t_2 & s_2 & 1 & \\ & & t_3 & s_3 & \ddots \\ & & & \ddots & \ddots \end{bmatrix}.$$

Examples of Catalan-like numbers

- Bell numbers B_n: $\mathfrak{s} = \mathfrak{t} = (1,2,3,\ldots)$.
- Motzkin numbers M_n: $\mathfrak{s} = \mathfrak{t} = (1,1,1,\ldots)$.
- Fine numbers F_n: $\mathfrak{s} = (0,2,2,\ldots)$, $\mathfrak{t} = (1,1,1,\ldots)$.
- Catalan numbers C_n: $\mathfrak{s} = (1,2,2,\ldots)$, $\mathfrak{t} = (1,1,1,\ldots)$.
- Riordan numbers R_n: $\mathfrak{s} = (0,1,1,\ldots)$, $\mathfrak{t} = (1,1,1,\ldots)$.
- Factorial numbers $n!$: $\mathfrak{s} = (1,3,5,\ldots)$, $\mathfrak{t} = (1,4,9,\ldots)$.
- Large Schröder numbers r_n: $\mathfrak{s} = (2,3,3,\ldots)$, $\mathfrak{t} = (2,2,2,\ldots)$.
- Little Schröder numbers S_n: $\mathfrak{s} = (1,3,3,\ldots)$, $\mathfrak{t} = (2,2,2,\ldots)$.
- Central Delannoy numbers D_n: $\mathfrak{s} = (3,3,3,\ldots)$, $\mathfrak{t} = (4,2,2,\ldots)$.
- Central trinomial coefficients T_n: $\mathfrak{s} = (1,1,1,\ldots)$, $\mathfrak{t} = (2,1,1,\ldots)$.
- Central binomial coefficients $\binom{2n}{n}$: $\mathfrak{s} = (2,2,2,\ldots)$, $\mathfrak{t} = (2,1,1,\ldots)$.

Main result

Theorem

The Catalan-like numbers form a HM sequence if and only if all t_k are nonnegative.

Proof. By the definition and lattice path proofs, we have

$$a_{m+n} = \sum_{k\geq 0} a_{m,k} a_{n,k} T_k \iff H = ATA^t$$

where $T = \mathrm{diag}(T_0, T_1, \ldots)$ and $T_0 = 1$, $T_k = t_1 t_2 \cdots t_k$ for $k \geq 1$.
Hankel determinants of the Catalan-like numbers a_n:

$$\det[a_{i+j}]_{0\leq i,j\leq n} = T_0 T_1 \cdots T_n = t_1^n t_2^{n-1} \cdots t_{n-1}^2 t_n.$$

□

∞-log-convex and Stieltjes moment sequences

For $\alpha = (a_n)_{n\geq 0}$, define $L(\alpha) = (b_n)_{n\geq 0}$ by

$$b_n = a_n a_{n+2} - a_{n+1}^2.$$

Then α is log-convex $\Longleftrightarrow$ $L(\alpha)$ is nonnegative.

Definition (William Y. C. Chen, arXiv: 0806.3392)

Call α infinitely log-convex if $L^m(\alpha)$ is nonnegative for all $m \geq 1$.

Conjecture (Chen and Xia, PAMS, 2011)

The sequence of the large Schröder numbers is ∞-log-convex.

Theorem (Wang and Zhu, Adv. in Appl. Math., 2016)

The Stieltjes moment sequence is ∞-log-convex.

∞-convex and Hamburger moment sequences

For $\alpha = (a_n)_{n\geq 0}$, define $\mathscr{L}(\alpha) = (c_n)_{n\geq 0}$ by

$$c_n = a_n + a_{n+2} - 2a_{n+1}.$$

Then α is convex $\Longleftrightarrow$ $\mathscr{L}(\alpha)$ is nonnegative.

- $\mathscr{L} = \Delta^2$.

Definition

Call α ∞-convex if $\mathscr{L}(\alpha)$ is nonnegative for all $m \geq 1$.

Theorem

The Hamburger moment sequence is ∞-convex.

Further work

The Apéry numbers

$$A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \quad n = 0, 1, 2, \ldots,$$

were introduced by Apéry in his famous proof to the irrationality of

$$\zeta(3) = \sum_{n \geq 1} n^{-3}.$$

Conjecture (Chen and Xia, PAMS, 2011)

$(A_n)_{n \geq 0}$ *is* ∞*-log-convex.*

Conjecture (Sokal, MIT, 2014)

$(A_n)_{n \geq 0}$ *is SM.*

Thank you for your attention!