Fourth international symposium on Riordan arrays and related topics.

Universidad Complutense de Madrid. Spain.
Facultad de Ciencias Matemáticas.
Aula Miguel de Guzmán

Book of Abstracts

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Contents

Chapter 1

Keynote speakers

1.1 Froilán M. Dopico

Polynomial eigenvalue problems: linearizations and global backward error analysis

Froilán M. Dopico

Departamento de Matemáticas. Universidad Carlos III de Madrid (Spain)

(joint work with Piers Lawrence, Javier Pérez, and Paul Van Dooren)

Given an square matrix $P(\lambda)$ whose entries are complex scalar polynomials in the variable λ, or, equivalently, a polynomial in λ whose coefficients are complex square matrices, the associated polynomial eigenvalue problem consists of finding complex numbers λ_0 and complex vectors v such that $P(\lambda_0)\, v = 0$. The numerical solution of polynomial eigenvalue problems arises in many applications in Engineering, Mechanics, Control, Computer Aided-Geometric Design, and, more recently, in Network Analysis. As a consequence, algorithms for the numerical solution of polynomial eigenvalue problems have received considerable attention in the last decade, both for medium size matrices as for large scale problems. Although different approaches are available in the literature for solving this problem, the one used most frequently is via linerizations, i.e., by transforming first the problem into a linear problem $L(\lambda_0)y = 0$, where all the entries of the matrix $L(\lambda)$ are linear polynomials in λ, and then applying a well established algorithm for linear (or generalized) eigenvalue problems. Despite of impressive achievements, many problems remain open on the solution of polynomial eigenvalue problems solved via linearizations as, for instance, a proper treatment of error and perturbation analyses. In this scenario, we revise in this talk at a expository level a number of recent results on linearizations and backward error analysis of polynomial eigenvalue problems solved via linearizations.

1.2 Anthony G. O'Farrell

Reversibility

Anthony G. O'Farrell

Maynooth University (National University of Ireland Maynooth)

Recently, workers on the Riordan group and its subgroups have taken an interest in reversibility. I will give some background information on the origins of reversibility in classical dynamics, and the many contexts in which it appears. I will point to some open problems.

1.3 Luis Verde-Star

Some ideas, results, and problems from the theory of infinite matrices

Luis Verde-Star

Universidad Autónoma Metropolitana, México

Infinite matrices have been studied for a long time, in different settings and using algebraic and analytic approaches, and they have been applied is several areas such as Quantum Mechanics, Probability, Summability theory, Group representations, and Combinatorics. I will present several ingredients for an algebraic (convergence free) theory of infinite matrices that can be applied in Combinatorics, Riordan arrays, and Orthogonal Polynomials. Some of the topics that will be discussed are the following

The different settings, bi-infinite matrices, multimatrices.

Matrices as representations of linear operators.

Lagrange inversion in several versions.

The Pincherle derivative.

Generating functions of matrices and their convolution using residues.

Matrices as formal Laurent series over a ring of sequences.

Applications to Riordan arrays and Orthogonal Polynomials.

Chapter 2

Invited speakers

2.1 Tian-Xiao He

Double Riordan Arrays and their compressions

Tian-Xiao He

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Inspired by the Fibonacci tree shown in the recent book, Catalan Numbers, by Richard Stanley, we present a combinatorial way to construct a type of lower triangle matrices related to the double Riordan arrays by using the succession rule, an ECO technique. We call those lower triangle matrices the compressions of the double Riordan arrays.

The Pascal-Fibonacci triangle

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 2 & 1 & 0 & 0 & \dots \\ 1 & 1 & 3 & 2 & 1 & 0 & \dots \\ 1 & 1 & 4 & 3 & 3 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

constructed from the Fibonacci tree shown in the Stanley's book is an example of the compressions of the double Riordan arrays.

We give sequence characterizations of the double Riordan arrays, subgroups of the double Riordan group, and the compressions of the double Riordan arrays. Some connections of the Riodan arrays and the double Riordan arrays as well as their compressions are established. The properties of the double Riordan arrays and their compressions such as those related to their P matrices and stabilizers are studied. The applications of the the linear map induced by the double Riordan arrays and their compressions are presented.

2.2 Donatella Merlini

Riordan arrays methods for combinatorial identities generation

Donatella Merlini

Dipartimento di Statistica, Informatica, Applicazioni, Università di Firenze, Italy

(Joint work with Renzo Sprugnoli)

Riordan arrays are recognized as a powerful tool for proving combinatorial identities and to perform combinatorial inversion in a constructive way, two problems for which the properties of Riordan arrays seem to be optimal. In fact, these properties are derived from the Fundamental Rule of Riordan Arrays (FRRA), which asserts: if $D = \mathcal{R}(d(t),\ h(t))$ is a Riordan array and $(f_k)_{k\in\mathbf{N}_S}$ is any sequence with $f(t) = \mathcal{G}(f_k)$, then:

$$\sum_{k=0}^{n} d_{n,k} f_k = [t^n] d(t) f(h(t)),$$

which allows us to compute a combinatorial sum by performing a suitable transformation of $f(t)$ and a coefficient extraction.

Unfortunately, it is not always possible to find the explicit generating function $f(t)$ or extract the coefficient of t^n. In the latter case we can often find an asymptotic formula, which may be as useful as an exact one, but other approaches can be devised.

In this talk we wish to present some generating techniques, derived by the theory of Riordan arrays, which produce lists of identities, giving an idea of what we can obtain by these methods.

After showing many approaches of direct application of the FRRA, another method we wish to show is derived from the A-sequence. Let $D = \mathcal{R}(d(t),\ h(t))$ be any Riordan array. It is well-known that a relation exists connecting the functions $h(t)$ and $A(t)$, the A-sequence of the array: $h(t) = tA(h(t))$. Passing to elements:

$$d_{n+1,k+1} = \sum_{j=0}^{n} a_j d_{n,k+j},$$

which shows that every interior item of the array is the closure of a sum. For example, if we set $C = \mathcal{R}\left(\frac{1-\sqrt{1-4t}}{2t}, \frac{1-\sqrt{1-4t}}{2}\right)$, C represents the Catalan triangle, the A-sequence of which corresponds to $A(t) = 1/(1-t)$, and the corresponding identity is:

$$\frac{k+2}{n+2}\binom{2n-k+1}{n-k} = \sum_{j=0}^{n} \frac{k+j+1}{n+1}\binom{2n-k-j}{n-k-j}.$$

Riordan arrays can be extended from $-\infty$ to $+\infty$ and this extended structures are called *bi–infinite recursive matrices*. Looking at these matrices upside–down, we see that the role of the function $A(t)$ has passed to $h(t)/t$ and *vice versa*, so that the two identities, where $a_j^{(m)} = [t^j]A(t)^m$ and $h_{j+m}^{(m)} = [t^j](h(t)/t)^m$:

$$d_{n+m,k+m} = \sum_{j=0}^{n-k} a_j^{(m)} d_{n,k+j}, \qquad d_{n+m,k+m} = \sum_{j=0}^{n-k} h_{j+m}^{(m)} d_{n-j,k}$$

hold true $\forall n, k, m \in \mathbb{Z}$. These formulas contain three integer parameters, so we can derive an indefinite number of (lists of) identities. The main drawback of this approach is that, given a specific identity, it is a matter of luck to find in lists the desired formula. Looking from a more positive angle, if you find your identity or meet some formulas confirming your ideas, you have solved your problem without needing any proof, implicit in the list. With this in mind, various authors have produced lists of combinatorial interest.

Often, combining identities obtained with Riordan arrays can give other interesting identities: for example, by using a family of Riordan arrays which transform arithmetic progressions into geometric progressions we find:

$$\sum_{k=0}^{n} \frac{2k+1}{n+k+1}\binom{2n}{n-k}\left(F_{4k+2} - (2k+1) + (-1)^k\right) = 9^n - 4^n + \delta_{n,0}$$

and by making the difference $\mathcal{R}_1 - \mathcal{R}_2$ between two particular Riordan arrays $\mathcal{R}_1$ and $\mathcal{R}_2$ in this family we can transform the sequence $2k + \frac{\sqrt{5}}{2}$ into the sequence of Fibonacci numbers $F_n = \frac{\phi^n - \bar{\phi}^n}{\sqrt{5}}$.

Other interesting identities arise while computing the path length of generating trees related to Riordan arrays. The Catalan generating tree, for example, defined by the rule

$$\begin{cases} root: & (1) \\ rule: & (k) \quad \to (1)\cdots(k)(k+1) \end{cases}$$

translates into the following identity, giving the total internal path length up to level n in the tree:

$$\sum_{i=0}^{n}\sum_{r=0}^{i}\frac{(r+1)^2}{2i-r+1}\binom{2i-r+1}{i-r}\frac{2+r}{2+r+2(n-i)}\binom{2+r+2(n-i)}{n-i}=\frac{1}{2}4^{n+1}-\frac{1}{2}\binom{2(n+1)}{n+1}.$$

This is again a not trivial application of the fundamental theorem with $\mathcal{R}\left(\frac{1-\sqrt{1-4t}}{2t},\frac{1-\sqrt{1-4t}}{2}\right)$.

As already observed, another interesting problem which can be handled with Riordan arrays is combinatorial inversion, which can be treated also in the *non proper* case $h(t)=t^s v(t)$, with $s\geq 1, v(0)\neq 0$, and allows us to find the following kind of formulas

$$a_n=\sum_{k=0}^{\lfloor n/s\rfloor} d_{n,k}b_k,\qquad b_n=\sum_{k=0}^{ns} d^*_{n,k},a_k$$

where $d^*_{n,k}$ is the (left) inverse of $d^*_{n,k}$.

Finally, by using Riordan arrays we are able to find identities for any *C-finite sequence*, that is, any sequence defined by a homogeneous linear recurrence relation with constant coefficients. For example, with this general approach we find the following formulas for Fibonacci and trinomial numbers:

$$F_n=\sum_{k=0}^{n-1}\binom{(n-1+k)/2}{(n-1-k)/2},\quad T_n=\sum_{k=0}^{n}\sum_{j=0}^{(n-k-1)/2}\binom{n-k-j-1}{n-k-2j-1}\binom{j}{k}.$$

In this talk we survey all these general methods of combinatorial identities generation, in which the fundamental theorem and the A-sequence paly the role of main actors.

2.3 José L. Ramírez

Lattice Paths, k-Bonacci Numbers and Riordan Arrays

José L. Ramírez

Universidad Nacional de Colombia, Colombia

(Joint work with Victor Sirvent)

This talk concerns paths counted by Riordan arrays whose sum on the rising diagonal is the k-bonacci sequence. In particular, we introduce a family of weighted lattice paths, whose step set is

$$\{H = (1,0), V = (0,1), D_1 = (1,1), \ldots, D_{m-1} = (1, m-1)\}.$$

From these lattice paths, we define a family of Riordan arrays whose sum on the rising diagonal is the k-bonacci sequence. This construction generalizes the Pascal and Delannoy Riordan arrays [1], whose sum on the rising diagonal is the Fibonacci and tribonacci sequence, respectively. From this family of Riordan arrays we introduce a generalized k-bonacci polynomial sequence, and we give a lattice path combinatorial interpretation of these polynomials [2]. We also show a combinatorial interpretation for the inverse of these Riordan arrays [3]. These results generalize the interpretation for the Schröder matrices of the first and the second kind [4].

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Chapter 3

Contributed speakers

3.1 José Agapito Ruiz

Combinatorics of a generalized Narayana identity

José Agapito Ruiz

University of Lisbon, Portugal

(Joint work with Â. Mestre, P. Petrullo and M. M. Torres)

The Narayana identity is a well-known formula that expresses the classical Catalan numbers as sums of the ordinary Narayana numbers. In this talk I will present a generalization of the Narayana identity for a family of Riordan arrays that includes the array of ballot numbers, the classical Catalan triangle and several other generalized Catalan triangles studied recently. I will give a combinatorial description for such an identity based on non-crossing partitions and say a few words on some other related aspects.

3.2 Paul Barry

Increasing trees and exponential Riordan arrays

Paul Barry

Waterford Institute of Technology, Waterford

An increasing tree is a labelled rooted tree in which labels along any branch from the root go in increasing order. Such trees can represent permutations, data structures in computer science, and probabilistic models in diverse applications [1]. Associated to such trees is the degree weight generating function $\phi(x)$, where

$$\phi(x) = \sum_{n=0}^{\infty} \phi_n \frac{x^n}{n!},$$

and where there are ϕ_n sorts of nodes of out-degree n in the tree. To each such generating function $\phi(x)$, we associate the exponential Riordan array whose production matrix has its bivariate generating function given by

$$e^{xy}(\phi'(x) + y\phi(x)).$$

In this context, exponential Riordan arrays become a useful bridge between increasing trees and other objects of combinatorial interest. Many of the resulting Riordan arrays are already studied in other contexts. Generalizations are possible. For instance, it is also possible to extend these links to bi-labelled trees.

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3.3 Xi Chen

Asymptotical normality of combinatorial sequences

Xi Chen

Dalian University of Technology, Dalian, P.R. China

(Joint work with Jianxi Mao and Yi Wang)

Let $a(n,k)$ be a double-indexed sequence of nonnegative numbers and $p(n,k) = a(n,k)/\sum_{j=0}^{n} a(n,j)$ the normalized probabilities. We say that $a(n,k)$ is asymptotically normal by a central limit theorem with mean μ_n and variance σ_n^2 if

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| \sum_{k\le\mu_n+x\sigma_n} p(n,k) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt \right| = 0.$$

Say that $a(n,k)$ is asymptotically normal by a local limit theorem on S if

$$\lim_{n\to\infty} \sup_{x\in S} \left| \sigma_n p(n, \lfloor \mu_n + x\sigma_n \rfloor) - \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right|.$$

In this talk, we survey the results about asymptotically normal sequences, as well as several methods and techniques for proving asymptotical normality. We also show that the Narayana numbers satisfy both central and local limit theorems, which in particular gives an affirmative answer to Shapiro's question in [1] about asymptotic normality of Narayana numbers.

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3.4 Gi-Sang Cheon

On Riordan graphs

Gi-Sang Cheon

Department of Mathematics, Sungkyunkwan University, Suwon 16419, Rep. of Korea

(Joint work with Ji-Hwan Jung and Seyed Ahmad Mojallal)

This talk is devoted to introducing new classes of graphs called *Riordan graphs*. There are many reasons to define new classes of graphs. These graphs can be used as computer networks with certain desired properties or to get useful information when designing algorithms to compute values of graph invariants. The Riordan graphs are found to exhibit a number of interesting properties which meet such reasons. We illustrate several examples including Pascal graphs, Catalan graphs, and Toeplitz graphs.

3.5 Sung-Tae Jin

A generalization of Riordan arrays and transition probability matrix

Sung-Tae Jin

Korea Institute for Advanced Study, Seoul 02455, Republic of Korea

(Joint work with Gi-Sang Cheon, Bong Dae Choi)

Riordan arrays are used extensively in many contexts as a combinatorial tool for solving enumeration problems. In this talk, we give a new generalization of Riordan arrays by using the concept of production matrix and transition probability matrix. The concept of transition probability matrix comes from Markov process. We will show how generalized Riordan arrays are used to solve the transient analysis of queueing models.

3.6 Ji-Hwan Jung

Horizontal and Vertical Recurrence Relations for Exponential Riordan Matrices and Their Applications

Ji-Hwan Jung

Department of Mathematics, Sungkyunkwan University, Suwon 16419, Rep. of Korea

(Joint work with Gi-Sang Cheon and Paul Barry)

In this talk, we show that an infinite lower triangular matrix $R = [r_{ij}]_{i,j\in\mathbb{N}_0}$ is an exponential Riordan matrix $R = \mathcal{E}(g, f)$ given by $\sum_{i\geq j} r_{ij} z^i/i! = g f^j/j!$ if and only if there exist both horizontal pair $\{h_n; \tilde{h}_n\}_{n\geq 0}$ and vertical pair $\{v_n; \tilde{v}_n\}_{n\geq 0}$ of the sequences such that

$$r_{n+1,k} = \tilde{h}_0 r_{n,k-1} + \sum_{j=0}^{n-k} \left((k+1)^{\overline{j}} h_j + k^{\overline{j+1}} \tilde{h}_{j+1} \right) r_{n,k+j}, \quad (n \geq k \geq 0)$$

and

$$r_{n,k-1} = \tilde{v}_0 r_{n+1,k} + \sum_{j=0}^{n-k} \left(n^{\underline{j}} v_j + n^{\underline{j+1}} \tilde{v}_{j+1} \right) r_{n-j,k}, \quad (n \geq k \geq 1)$$

where $x^{\overline{n}}$ and $x^{\underline{n}}$ are the rising and falling factorials of $x \geq 0$ defined by

$$x^{\overline{n}} = x(x+1)\cdots(x+n-1) \text{ and } x^{\underline{n}} = x(x-1)\cdots(x-n+1), \quad n \geq 1$$

with $x^{\overline{0}} = x^{\underline{0}} = 1$. Applying this result, we obtain that if the horizontal and vertical pairs of an exponential Riordan matrix are identical then the matrix is involution. In addition, this concept can be applied to obtain the determinants of the production matrix and some conditions for d-orthogonality of the Sheffer polynomial sequences.

3.7 Hana Kim

Riordan matrices related to Dirichlet series

Hana Kim

National Institute for Mathematical Sciences, Republic of Korea

(Joint work with Gi-Sang Cheon (Sungkyunkwan University, Republic of Korea))

The Riemann hypothesis states that all nontrivial zeros of the Riemann zeta function $\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$ lie on the critical line $\mathrm{Re}(s) = \frac{1}{2}$. In 1977, Redheffer introduced the $n \times n$ matrix R_n whose (i, j)-entry is 1 if either $i = 0$ or $j + 1$ divides $i + 1$; and is 0 otherwise for $0 \leq i, j \leq n - 1$, which is called the *Redheffer matrix*. It is known that the Riemann hypothesis is true if and only if

$$\det R_n = O(n^{1/2+\epsilon})$$

for all positive ϵ. In this talk, we introduce an infinite family of matrices using Riordan matrices in which each has the same determinant as that of the Redheffer matrix but may have different eigenvalues. We find the generating function for the characteristic polynomials of the new matrices, and explore several examples that reveal interesting spectrum.

The speaker is supported by National Institute of Mathematical Sciences (A23100000) and the National Research Foundation of Korea (NRF-2016R1C1B1014356).

3.8 Huyile Liang

Stieltjes and Hamburger moment sequences in combinatorics

Huyile Liang

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We show that many well-known counting coefficients in combinatorics are Stieltjes (Hamburger) moment sequences in certain unified approaches and that Hamburger moment sequences are infinitely convex. We introduce the concept of the q-Hamburger moment sequence of polynomials and present some examples of such sequences of polynomials. We also suggest some problems and conjectures.

3.9 Ana Luzón

Banach Fixed Point Theorem and Lagrange Inversion Formula

Ana Luzón

Departamento de Matemática Aplicada. Universidad Politécnica de Madrid. Spain

Lagrange Inversion Formula (LIF) is an useful tool in combinatorics. Recently Ira Gessel in [1] gives a interesting survey about this topic. Donatella Merlini, Renzo Sprugnoli and Cecilia Verri told us when and how we should use Lagrange Inversion in [3]. In this talk I will show our approach to get LIF using Banach Fixed Point Theorem. See [2].

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3.10 Seyed A. Mojallal

The spectral properties of Riordan graphs

Seyed Ahmad Mojallal

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(Joint work with Gi-Sang Cheon)

Riordan arrays are infinite lower triangular matrices defined by two generating functions [1]. Considering an n x n Riordan matrix in modulo two, we define Riordan graph RGn of order n. In this talk, we study some spectral invariants of Riordan graphs such as the adjacency spectrum, Laplacian and signless Laplacian spectrums, nullity, inertia and determinant of Riordan graphs. Moreover, we show that our results for Riordan graphs are better than previous results for general graphs.

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3.11 Manuel A. Morón

On some bivariate delta-evolution equations

Manuel A. Morón

Departamento de Geometría y Topología. Universidad Complutense de Madrid. Spain

(Joint work with A. Luzón)

In this talk we describe an application of Rota and collaborators's ideas, about foundations on combinatorial theory, to the computation of solutions of some linear functional partial differential equations. To do that we give a dynamical interpretation of convolution families of polynomials.

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3.12 L. Felipe Prieto-Martínez

The Riordan group in the f-vector problem

Luis Felipe Prieto-Martínez

Universidad Autónoma de Madrid, Spain

(Joint work with Ana María Luzón Cordero and Manuel Alonso Morón)

Given a finite simplicial complex K, its *face vector*, or simply *f-vector*, is a sequence $(f_0, f_1, f_2 \ldots)$ where f_d is the number of faces of dimension d in K (see [7]). The f-vector problem is a very general open question that could be stated as follows: how is the set of f-vectors of the family of simplicial complexes satisfying a given topological property (such as being a polytopal sphere)? A lot of work has been done in this direction in the bibliography (see for example [1-6]).

It will be shown how the Riordan group appears in different ways in the study of the f-vector problem: Dehn-Sommerville equations are not other thing than an eigenvector problem for a Riordan involution, the f-vectors of some families of simplicial complexes placed as rows give rise to a Riordan matrix, bounds on the number of possible linear relations on the f-vector of certain families of simplicial complexes can be found using the First Fundamental Theorem of Riordan Arrays,... and much more.

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3.13 Louis Shapiro

Riordan group theory

Louis Shapiro

Mathematics Department, Howard University, Washington, DC 20059, USA

We will examine some connections of abstract group theory to various enumeration questions. For instance recall that $m = \frac{1-z-\sqrt{1-2z-3z^2}}{2z^2}$ is the generating function for the Motzkin numbers. As one example we will show that $\left(m, \frac{-m+\sqrt{4m-3m^2}}{2}\right)$ is an element of order two and look at the underlying combinatorial meaning of the coefficients. A second simpler example is $(C, -C^3)$ where C is the Catalan generating function.

3.14 Sheng-Liang Yang

Diagonal sums of Riordan arrays

Sheng-Liang Yang

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It is well known that the sums of binomial coefficients along the rising diagonals of Pascal matrix are the Fibonacci numbers, while the rising diagonal sums of the Catalan triangle $(C(t)^2, tC(t)^2)$ are the Fine numbers. This motivate us to introduce the following definition.

Definition Let m be a nonnegative integer and s a positive integer, and let $R = (r_{n,k})_{n,k\in\mathbb{N}}$ be a Riordan array. The (m,s)-diagonal polynomials of R are defined as

$$d_n^{(m,s)}(x) = \sum_{j\geq 0} r_{n-mj,sj}x^j,\ n = 0, 1, 2, \cdots,$$

For $m = 0$ and $s = 1$, we obtain the row polynomials of R, and for $m = 1$ and $s = 1$, we obtain the rising diagonal polynomials. In this talk, we will present some recurrence relations and determinant formula for the (m,s)-diagonal polynomial sequence of Riordan arrays.

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