Lattice Paths, \(k\)-Bonacci Numbers and Riordan Arrays

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4th International Symposium on Riordan Arrays and Related Topics
Universidad Complutense de Madrid
Madrid - Spain
July - 2017
Outline

1. Lattice Path - Pascal Matrix - Delannoy Matrix
2. \((a, b, c, d)\)–weighted paths
3. The General Case
4. Schröder Matrix
5. Some additional comments.
Lattice Path – Riordan Arrays

Lattice Path – Riordan Arrays

Basic Problem

How many lattice paths are there from (0,0) to (n, m)?
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How many lattice paths are there from $(0, 0)$ to $(n, m)$?

$$D_1(n, k) = \binom{n + m}{n} = \binom{n + m}{m}$$
Pascal Matrix - Lattice Paths

\[ \mathcal{H}_1 := \begin{bmatrix} h^{(1)}_{n,k} \end{bmatrix}_{n,k \in \mathbb{N}}, \text{ where} \]

\[ h^{(1)}_{n,k} = \begin{cases} D_1(n - k, k) = \binom{n}{k}, & \text{if } n \geq k; \\ 0, & \text{if } n < k. \end{cases} \]

\[ \mathcal{H}_1 = \left( \frac{1}{1 - z}, \frac{z}{1 - z} \right) = \begin{bmatrix} 1 \\ 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \\ 1 & 5 & 10 & 10 & 5 & 1 \\ \vdots \end{bmatrix} \]

\( \mathcal{H}_1 \) is the Pascal matrix.
\[ F_n = \sum_{\ell=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-\ell-1}{\ell} \] is the number of lattice path from \((0,0)\) to \((n-1-2k, k)\), for \(k = 0, 1, \ldots, \lfloor (n-1)/2 \rfloor\).
Problem 2

How many \((a, b, c)\)-weighted lattice paths are there from \((0,0)\) to \((n, m)\)?
Generalized Delannoy number

**Generalized Delannoy matrix:** $\mathcal{H}_2(a, b, c) = [d_{n,k}]_{n,k \in \mathbb{N}}$, where

$$d_{n,k} = \begin{cases} 
D_2^*(n - k, k), & \text{if } n \geq k; \\
0, & \text{if } n < k.
\end{cases}$$

$$\mathcal{H}_2(a, b, c) = \left( \frac{1}{1 - bz}, z \frac{a + cz}{1 - bz} \right)$$

$$= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
b & a & 0 & 0 & 0 & 0 \\
b^2 & c + 2ab & a^2 & 0 & 0 & 0 \\
b^3 & b(2c + 3ab) & a(2c + 3ab) & a^3 & 0 & 0 \\
b^4 & b^2(3c + 4ab) & c^2 + 6abc + 6a^2b^2 & a^2(3c + 4ab) & a^4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}$$

**GF Row Sum:** $\frac{1}{1-(a+b)z-cz^2}$

Delannoy Matrix

**Delannoy Matrix:**

\[ \mathcal{H}_2(1, 1, 1) = \left( \frac{1}{1 - z}, \frac{z(1 + z)}{1 - z} \right) \]

\[
= \begin{bmatrix}
1 \\
1 & 1 \\
1 & 3 & 1 \\
1 & 5 & 5 & 1 \\
1 & 7 & 13 & 7 & 1 \\
1 & 9 & 25 & 25 & 9 & 1 \\
\vdots
\end{bmatrix}
\]

**GF Row Sum:**

\[ \frac{1}{1 - 2z - z^2} \]
Delannoy Matrix

The sum of the elements on the rising diagonal is the **tribonacci sequence** $t_n$. 

\[
\begin{array}{ccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 2 & 4 & 7 & 13 & \\
1 & 3 & 1 & 1 & 1 & 1 & 1 \\
1 & 5 & 5 & 1 & 1 & 1 & 1 \\
1 & 7 & 13 & 7 & 1 & 1 & 1 \\
1 & 9 & 25 & 25 & 9 & 1 & 1 \\
\end{array}
\]
Delannoy Matrix

\[ T^{(a,b,c)}_{n+1,q,r,p} := \text{the sum on the diagonal of direction } (r, q), \text{ i.e.,} \]

\[ T^{(a,b,c)}_{n+1,q,r,p} = \left\lfloor \frac{n-p}{q+r} \right\rfloor \sum_{k=0}^{n} d(n - qk, p + rk). \]

Direction \((1, q)\).

Theorem

The sequence \( T_{n,q} := T^{(a,b,c)}_{n,q,1,0} \) satisfies the following linear recurrence relation

\[ T_{n,q} = bT_{n-1,q} + aT_{n-q-1,q} + cT_{n-q-2,q}, \]

with the initial values \( T_{1,q} = 1 \), and

\( T_{0,q} = T_{1,q} = \cdots = T_{n-q-2,q} = 0. \)
Delannoy Matrix

\[ T_{n+1,q,r,p}^{(a,b,c)} := \text{the sum on the diagonal of direction } (r, q), \text{ i.e.,} \]

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Delannoy Matrix

Proof:

\[ (0, 0) \]

Corollary

The generating function of the sequence \( \{ T_{n+1, q} \}_{n \geq 0} \) is

\[
\sum_{n \geq 0} T_{n+1, q} z^n = \frac{1}{1 - b z - a z q + 1 + c z q + 2}.
\]
Delannoy Matrix

Proof:

Corollary

The generating function of the sequence \( \{T_{n+1,q}\}_{n \geq 0} \) is

\[
\sum_{n \geq 0} T_{n+1,q} z^n = \frac{1}{1 - bz - azq+1 + czq+2}.
\]
**Delannoy Matrix**

**General Case:** this corresponds to the finite sequences lying over finite transversals of Delannoy matrix.

**Theorem**

The sequence \( T_{n,q,r,p} := T_{n,q,r,p}^{(a,b,c)} \) satisfies the following linear recurrence relation

\[
\sum_{s=0}^{r} (-b)^s \binom{r}{s} T_{n-s,q,r,p} = \sum_{s=0}^{r} a^s c^s \binom{r}{s} T_{n-r-q-s,q,r,p}.
\]

**Theorem**

The generating function of the sequence \( \{T_{n+1,q,r,p}\}_{n \geq 0} \) is given by

\[
\sum_{n \geq 0} T_{n+1,q,r,p}z^n = \frac{(1 - bz)^{r-p-1}(a + cz)^p z^p}{(1 - bz)^r - (a + cz)^r z^{q+r}}.
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\]
Example

For $r = q = 1$ and $p = 0$, we have the tribonacci sequence $(T_n)_{n \geq 0} = (0, 1, 1, 2, 4, 7, 13, 24, \ldots)$. Therefore, we have the explicit relation

$$T_{n+1} = \sum_{k=0}^{[n/2]} \sum_{j=0}^{k} \binom{j}{k} \binom{n-j-k}{k}.$$
Particular Case

\[ \mathcal{H}_2(1,0,1) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \]

Row Sum = Fibonacci Sequence

GF Row Sum:

\[ \frac{1}{1 - z - z^2} \]
Combinatorial Interpretation - Fibonacci Numbers

\[ F_{n+1} = \sum_{k=0}^{n} \omega_{k, n-k} \]

\( \omega_{k, n-k} := \) is the sum of weights of paths from \((0, 0)\) to \((k, n-k)\) using the steps:

\((1, 0), \quad (1, 1)\)

Combinatorial Interpretation - Fibonacci Numbers

Example

\[ F_5 = 5 \]
Summarizing

- **Pascal Matrix** – Sum of the elements on the rising diagonal = Fibonacci numbers.
- **Delannoy Matrix** – Sum of the elements on the rising diagonal = Tribonacci numbers.
- **XXXXX Matrix** (Riordan array) – Sum of the elements on the rising diagonal = \(k\)-bonacci numbers.

What family of steps are required to find a Riordan array, such that the sum of the elements on its main diagonal (or on its row) is the \(k\)-bonacci sequence \(F_n^{(k)}\)?
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What family of steps are required to find a Riordan array, such that the sum of the elements on its main diagonal (or on its row) is the \( k \)-bonacci sequence \( F_n^{(k)} \)?
Combinatorial Interpretation?

- **k-bonacci numbers** The $k$-bonacci numbers are defined by the recurrence

\[
\mathcal{F}_n^{(k)} = \mathcal{F}_{n-1}^{(k)} + \mathcal{F}_{n-2}^{(k)} + \cdots + \mathcal{F}_{n-k}^{(k)}, \quad n \geq 1,
\]

with initial values $\mathcal{F}_{-1}^{(k)} = \mathcal{F}_{-2}^{(k)} = \cdots = \mathcal{F}_{-(k-1)}^{(k)} = 0$ and $\mathcal{F}_0^{(k)} = 1$.

$k = 3$, **tribonacci**: $1, 1, 2, 4, 7, 13, 24, \ldots$

- **k-bonacci polynomials**

\[
\left[ \mathcal{F}_n^{(k)} \right](x) = x^{k-1} \mathcal{F}_{n-1}^{(k)}(x) + x^{k-2} \mathcal{F}_{n-2}^{(k)} + \cdots + \mathcal{F}_{n-k}^{(k)}, \quad n \geq 1.
\]
Combinatorial Interpretation?

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with initial values $F_{-1}^{(k)} = F_{-2}^{(k)} = \cdots = F_{-(k-1)}^{(k)} = 0$ and

$F_0^{(k)} = 1$.

*k = 3*, tribonacci: 1, 1, 2, 4, 7, 13, 24, \ldots

- **k-bonacci polynomials**

$$F_n^{(k)}(x) = x^{k-1}F_{n-1}^{(k)}(x) + x^{k-2}F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)}, \quad n \geq 1.$$
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with initial values \( \mathcal{F}_{-1}^{(k)} = \mathcal{F}_{-2}^{(k)} = \cdots = \mathcal{F}_{-(k-1)}^{(k)} = 0 \) and \( \mathcal{F}_0^{(k)} = 1 \).

\( k = 3 \), **tribonacci**: 1, 1, 2, 4, 7, 13, 24, …

- **k-bonacci polynomials**.

\[
\mathcal{F}_n^{(k)}(x) = x^{k-1}\mathcal{F}_{n-1}^{(k)}(x) + x^{k-2}\mathcal{F}_{n-2}^{(k)} + \cdots + \mathcal{F}_{n-k}^{(k)}, \quad n \geq 1,
\]
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\((a, b, c, d)\)-weighted paths

Let \(\mathbb{M}_3(n, k)\) denote the set of \((a, b, c, d)\)-weighted paths from the point \((0, 0)\) to the point \((k, n)\), with step set \(S_3 = \{H = (1, 0), V = (0, 1), D_1 = (1, 1), D_2 = (1, 2)\}\), where each step is labelled with weights \(a, b, c\) and \(d\), respectively.

**Example**
$(a, b, c, d)$-weighted paths

\[ D_3(n, k) = a D_3(n-1, k) + b D_3(n, k-1) + c D_3(n-1, k-1) + d D_3(n-1, k-2), \]

with $k \geq 2$, $n \geq 1$ and initial conditions $D_3(0, k) = b^k$ and $D_3(n, 0) = a^n$. 
(a, b, c, d)-weighted paths

Theorem

The number of (a, b, c, d)-lattice paths is given by

\[
D_3(n, k) = \sum_{j=0}^{n} \sum_{l=0}^{j} \binom{n}{j} \binom{j}{l} \left( \begin{array}{c} n + k - 2j + l \\ k - 2j + l \end{array} \right) a^{n-j} c^l d^{j-l} b^{k-2j+l}.
\]

= \sum_{j=0}^{n} \sum_{l=0}^{n-j} \binom{n}{j} \binom{n-j}{l} \left( \begin{array}{c} k - n + 2j + l \\ n \end{array} \right) a^j c^l d^{n-j-l} b^{k-2n+2j+l}.

Proof.

For \( n \geq 1 \), let \( \mathcal{W}_n^{(3)}(z) := \sum_{i=0}^{\infty} D_3(n, i)z^i \).

Then

\[
\mathcal{W}_n^{(3)}(z) = a\mathcal{W}_{n-1}^{(3)}(z) + bz\mathcal{W}_n^{(3)}(z) + cz\mathcal{W}_{n-1}^{(3)}(z) + dz^2\mathcal{W}_n^{(3)}(z).
\]
\((a, b, c, d)\)-weighted paths

Proof.

\[ \mathcal{W}^{(3)}_n(z) = \left( \frac{a + cz + dz^2}{1 - bz} \right) \mathcal{W}^{(3)}_{n-1}(z) = \left( \frac{a + cz + dz^2}{1 - bz} \right)^n \mathcal{W}^{(3)}_0(z) \]

\[ = \left( \frac{a + cz + dz^2}{1 - bz} \right)^n \frac{1}{1 - bz} = \frac{(a + cz + dz^2)^n}{(1 - bz)^{n+1}}. \]

Therefore, by the binomial theorem

\[ [z^k] \mathcal{W}^{(3)}_n = [z^k] \left( (a + cz + dz^2)^n (1 - bz)^{-(n+1)} \right) \]

\[ = \sum_{j=0}^{n} \sum_{l=0}^{j} \binom{n}{j} \binom{j}{l} \binom{n + k - 2j + l}{k - 2j + l} a^{n-j} c^j d^{j-l} b^{k-2j+l}. \]
(a, b, c, d)-weighted paths

Definition
Let \( H_3 \) := \( H_3(a, b, c, d) := \left[ d_{n,k}^{(3)} \right]_{n,k \in \mathbb{N}} \), where

\[
d_{n,k}^{(3)} = \begin{cases} 
D_3(n - k, k), & \text{if } n \geq k; \\
0, & \text{if } n < k.
\end{cases}
\]

Theorem
The infinite triangular array \( H_3(a, b, c, d) \) has a Riordan array expression given by

\[
H_3 = H_3(a, b, c, d) = \left( \frac{1}{1 - bz}, \frac{a + cz + dz^2}{1 - bz} \right)
\]
(a, b, c, d)-weighted paths

Definition
Let $\mathcal{H}_3 := \mathcal{H}_3(a, b, c, d) := \left[ d^{(3)}_{n,k} \right]_{n,k \in \mathbb{N}}$, where

$$d^{(3)}_{n,k} = \begin{cases} \mathcal{D}_3(n-k, k), & \text{if } n \geq k; \\ 0, & \text{if } n < k. \end{cases}$$

Theorem
The infinite triangular array $\mathcal{H}_3(a, b, c, d)$ has a Riordan array expression given by

$$\mathcal{H}_3 = \mathcal{H}_3(a, b, c, d) = \left( \frac{1}{1 - bz}, z \frac{a + cz + dz^2}{1 - bz} \right).$$
(**a, b, c, d**)-weighted paths

**Proposition**

Let $A_3(z)$ be the generating function for the rows sums of the Riordan array $\mathcal{H}_3$. Then

\[
A_3(z) = \frac{1}{1 - (a + b)z - cz^2 - dz^3}.
\]

- Let $a + b = p(x)$, $c = q(x)$, $d = r(x)$.
- Let $\mathcal{F}_n^{(3)}(x)$ be the $n$-th row sum of $\mathcal{H}_3$.

If $p = x^2$, $q = x$ and $r = 1$, we get the tribonacci polynomials.
(a, b, c, d)-weighted paths

Proposition

Let $A_3(z)$ be the generating function for the rows sums of the Riordan array $\mathcal{H}_3$. Then

$$A_3(z) = \frac{1}{1 - (a + b)z - cz^2 - dz^3}.$$ 

Let $a + b = \overline{p}(x)$, $c = \overline{q}(x)$, $d = \overline{r}(x)$.

Let $\mathcal{F}_n^{(3)}(x)$ be the $n$-th row sum of $\mathcal{H}_3$.

$$A_3(z) = \sum_{i=0}^{\infty} \mathcal{F}_i^{(3)}(x)z^i = \frac{1}{1 - \overline{p}(x)z - \overline{q}(x)z^2 - \overline{r}(x)z^3}.$$ 

If $p = x^2$, $q = x$ and $r = 1$, we get the tribonacci polynomials.
(a, b, c, d)-weighted paths

**Theorem**

For all integer $n \geq 3$, the polynomials $\mathcal{F}_n^{(3)}(x)$ satisfy the following recurrence:

$$
\mathcal{F}_n^{(3)}(x) = \overline{p}(x)\mathcal{F}_{n-1}^{(3)}(x) + \overline{q}(x)\mathcal{F}_{n-2}^{(3)}(x) + \overline{r}(x)\mathcal{F}_{n-3}^{(3)}(x),
$$

where $\mathcal{F}_0^{(3)}(x) = 1$, $\mathcal{F}_1^{(3)}(x) = \overline{p}(x)$, $\mathcal{F}_2^{(3)}(x) = \overline{p}^2(x) + \overline{q}(x)$. 
(a, b, c, d)-weighted paths

Theorem

For all integer $n \geq 0$, and for any polynomials $\overline{p}(x), \overline{q}(x), \overline{r}(x)$,

$$F_n^{(3)}(x) = \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \sum_{j=0}^{i} \binom{n-i-j}{i} \binom{i}{j} \overline{q}^{i-j}(x) \overline{r}^{j}(x) \overline{p}^{n-2i-j}(x).$$

$$F_n^{(3)}(x) = \sum_{0 \leq j \leq i \leq n} \binom{n-i}{i-j} \binom{i-j}{j} \overline{q}^{i-2j}(x) \overline{r}^{j}(x) \overline{p}^{n-2i+j}(x).$$
(a, b, c, d)-weighted paths - Combinatorial Interpretation

For any polynomial $F_{n}(x)$, we have

$$F_{n}(x) = \sum_{k=0}^{n} \omega_{n,k}, \ n \geq 0,$$

where $\omega_{n,k}(x) := h_{n,k}(x) = D_{3}(n - k, k)$ is the sum of weights of $(a(x), b(x), c(x), d(x))$-weighted paths from $(0, 0)$ to $(k, n - k)$ with step set

$$S_{3} = \{ H = (0, 0), V = (0, 1), D_{1} = (1, 1), D_{2} = (1, 2) \} ,$$

such that $a(x) + b(x) = \overline{p}(x)$, $c(x) = \overline{q}(x)$ and $d(x) = \overline{r}(x)$.
Tribonacci paths

The tribonacci polynomials $T_n(x)$:

$$T_n(x) = x^2 T_{n-1}(x) + xT_{n-2}(x) + T_{n-3}(x), \quad \text{for } n \geq 2.$$

$$\sum_{n=0}^{\infty} T_n(x)z^n = \frac{1}{1 - x^2z - xz^2 - z^3}.$$

$a(x) = x^2$, $b(x) = 0$, $c(x) = x$ and $d(x) = 1$ $H_3(x^2, 0, x, 1)$:

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & x^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & x & x^4 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2x^3 & x^6 & 0 & 0 & 0 & 0 \\
0 & 0 & 3x^2 & 3x^5 & x^8 & 0 & 0 & 0 \\
0 & 0 & 2x & 6x^4 & 4x^7 & x^{10} & 0 & 0 \\
0 & 0 & 1 & 7x^3 & 10x^6 & 5x^9 & x^{12} & \ddots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix} \cdot \begin{bmatrix}1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
\vdots
\end{bmatrix} = \begin{bmatrix}T_0(x) \\
T_1(x) \\
T_2(x) \\
T_3(x) \\
T_4(x) \\
T_5(x) \\
T_6(x) \\
\vdots
\end{bmatrix}.$$
Tribonacci paths

Example

\[ T_4(x) = x^8 + 3x^5 + 3x^2 = \sum_{i=0}^{4} \omega_{4,i}^{(3)}. \]  
Note that,  
\[ \omega_{0,4}^{(3)} = \omega_{1,3}^{(3)} = 0, \omega_{2,2}^{(3)} = 3x^2, \omega_{3,1}^{(3)} = 3x^5 \]  
and  
\[ \omega_{4,0}^{(3)} = x^8. \]
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The General Case

- Let $\vec{A}_s = (a_1, a_2, \ldots, a_s)$ be a vector of weights.
- We denote by $\mathbb{M}_m(n, k)$ the set of $\vec{A}_{m+1}$-weighted paths from the point $(0, 0)$ to the point $(k, n)$, with step set

$$S_m = \{ H = (1, 0), V = (0, 1), D_1 = (1, 1), \ldots, D_{m-1} = (1, m-1) \},$$

where each step is labelled with weights $a_1, a_2, \ldots, a_{m+1}$, respectively.
- $\mathcal{D}_m(n, k) = |\mathbb{M}_m(n, k)|$. 
The General Case

Lemma

The numbers $D_m(n, k)$ satisfy the following $(m + 1)$-term recurrence relation

$$D_m(n, k) = a_1D_m(n - 1, k) + a_2D_m(n, k - 1) + \sum_{j=1}^{m-1} a_{j+2}D_m(n - 1, k - j)$$

with $k \geq m - 1$, $n \geq 1$ and initial conditions $D_m(0, k) = a_k^2$ and $D_m(n, 0) = a_1^n$.

$S_m = \{ H = (1, 0), V = (0, 1), D_1 = (1, 1), \ldots, D_{m-1} = (1, m - 1) \}$,
Riordan Arrays and Generalized $k$-bonacci numbers

Theorem

The number of $\vec{A}_{m+1}$-lattice paths is given by

$$D_m(n, k) = \sum_{j_1=0}^{n} \sum_{j_2=0}^{n-j_1} \cdots \sum_{j_{m-1}=0}^{n-\sum_{j=1}^{m-2} j_i} \binom{n}{j_1} \binom{n-j_1}{j_2} \cdots \binom{n-\sum_{j=1}^{m-2} j_i}{j_{m-1}} \times \frac{(n+k-u)}{n} a_1^{j_1} a_3^{j_2} \cdots a_{m-1}^{j_{m-1}} a_{m+1}^{n-\sum_{i=1}^{m-1} j_i} a_2^{k-u},$$

where

$$u = (m-1)(n-j_1) + \sum_{i=2}^{m-1} (i-m)j_i.$$
Riordan Arrays and Generalized $k$-bonacci numbers

Definition
Let $\mathcal{H}_m := \mathcal{H}_m(a_1, a_2, \ldots, a_{m+1}) := \left[ d_{n,k}^{(m)} \right]_{n,k \in \mathbb{N}}$, where

$$d_{n,k}^{(m)} = \begin{cases} D_m(n - k, k), & \text{if } n \geq k; \\ 0, & \text{if } n < k. \end{cases}$$

Theorem
The infinite triangular array $\mathcal{H}_m$ has a Riordan array expression given by

$$\mathcal{H}_m = \left( \frac{1}{1 - a_2 z}, \frac{z a_1 + a_3 z + a_4 z^2 + \cdots + a_{m+1} z^{m-1}}{1 - a_2 z} \right).$$
Riordan Arrays and Generalized $k$-bonacci numbers

**Proposition**

The $k$-bonacci triangle $T_k$ defined by the following Riordan array

\[
T_k = \left( \frac{1}{1 - z}, z \frac{1 + z + \cdots + z^{k-1}}{1 - z} \right),
\]

satisfies that the sum on the rising diagonal is the $k$-bonacci sequence $F_n^{(k)}$, where $F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)}$. 

Riordan Arrays and Generalized $k$-bonacci numbers

**Proposition**

Let $A_m(z)$ be the generating function for the rows sums of the Riordan array $H_m$. Then

$$A_m(z) = \frac{1}{1 - (a_1 + a_2)z - a_3z^2 - \cdots - a_{m+1}z^m}.$$

Let $F_n^{(k)}(x)$ be the $n$-th row sum of $H_k$ (generalized $k$-bonacci polynomials):

$$A_k(z) = \sum_{i=0}^{\infty} F_i^{(k)}(x)z^i = \frac{1}{1 - \overline{p}_1(x)z - \overline{p}_2(x)z^2 - \cdots - \overline{p}_k(x)z^k},$$

where $\overline{p}_1(x) = a_1 + a_2$, and $\overline{p}_j(x) = a_{j+1}, j = 2, \ldots, k$. 
Riordan Arrays and Generalized $k$-bonacci numbers

**Proposition**

Let $A_m(z)$ be the generating function for the rows sums of the Riordan array $\mathcal{H}_m$. Then

$$A_m(z) = \frac{1}{1 - (a_1 + a_2)z - a_3 z^2 - \cdots - a_{m+1} z^m}.$$ 

Let $F_n^{(k)}(x)$ be the $n$-th row sum of $\mathcal{H}_k$ (generalized $k$-bonacci polynomials):

$$A_k(z) = \sum_{i=0}^{\infty} F_i^{(k)}(x) z^i = \frac{1}{1 - \overline{p}_1(x) z - \overline{p}_2(x) z^2 - \cdots - \overline{p}_k(x) z^k},$$

where $\overline{p}_1(x) = a_1 + a_2$, and $\overline{p}_j(x) = a_{j+1}, j = 2, \ldots, k.$
Riordan Arrays and Generalized $k$-bonacci numbers

Theorem

The polynomials $F_n^{(k)}(x)$ satisfy the following recurrence for $n \geq k$

$$F_n^{(k)}(x) = p_1(x)F_{n-1}^{(k)}(x) + p_2(x)F_{n-2}^{(k)}(x) + \cdots + p_k(x)F_{n-k}^{(k)}(x),$$

where $F_0^{(k)}(x) = 1, F_i^{(k)}(x) = 0$ for $i = -1, -2, \ldots, -(k - 1)$. 
Combinatorial Interpretation and Generalized $k$-bonacci numbers

For any polynomial $\mathcal{F}_n^{(m)}(x)$, we have

$$\mathcal{F}_n^{(m)}(x) = \sum_{k=0}^{n} \omega_{n,k}^{(m)}, \ n \geq 0,$$

where $\omega_{n,k}^{(m)}(x) := h_{n,k}^{(m)} = D_m(n - k, k)$ is the sum of weights of $\vec{A}_{m+1}$-weighted paths from $(0, 0)$ to $(k, n - k)$ with step set

$$S_m = \{ H = (0, 0), V = (0, 1), D_1 = (1, 1), D_2 = (1, 2), \ldots, D_{m-1} = (1, m-1) \},$$

such that $a_1(x) + a_2(x) = \overline{p}_1(x)$, and $a_i(x) = \overline{p}_{i-1}(x)$ for $i = 1, \ldots, m + 1$. 
Riordan Arrays and Generalized $k$-bonacci numbers

Example
The tetrabonacci numbers:

$$l_0 = 1, \quad l_1 = 1, \quad l_2 = 2, \quad l_3 = 4$$

$$l_{n+4} = l_{n+3} + l_{n+2} + l_{n+1} + l_n, \quad \text{for } n \geq 0.$$  

The tetrabonacci polynomials $R_n(x)$ are defined by the recurrence relation

$$R_0(x) = 1, \quad R_1(x) = x^3, \quad R_2(x) = x^6 + x^2, \quad R_3(x) = x^9 + 2x^5 + x,$$
$$R_{n+4}(x) = x^3 R_{n+3}(x) + x^2 R_{n+2}(x) + x R_{n+1}(x) + R_n(x), \quad \text{for } n \geq 0.$$  

If $a_1(x) = x^3, a_2(x) = 0, a_3(x) = x^2, a_4(x) = x$ and $a_5(x) = 1$, we obtain the tetrabonacci polynomials from the row sums of the Riordan array $\mathcal{H}_4(x^3, 0, x^2, x, 1)$. 
Riordan Arrays and Generalized $k$-bonacci numbers

Example

The tetrabonacci numbers:

\[ l_0 = 1, \quad l_1 = 1, \quad l_2 = 2, \quad l_3 = 4 \]
\[ l_{n+4} = l_{n+3} + l_{n+2} + l_{n+1} + l_n, \quad \text{for} \quad n \geq 0. \]

The tetrabonacci polynomials $R_n(x)$ are defined by the recurrence relation

\[ R_0(x) = 1, \quad R_1(x) = x^3, \quad R_2(x) = x^6 + x^2, \quad R_3(x) = x^9 + 2x^5 + x, \]
\[ R_{n+4}(x) = x^3R_{n+3}(x) + x^2R_{n+2}(x) + xR_{n+1}(x) + R_n(x), \quad \text{for} \quad n \geq 0. \]

If $a_1(x) = x^3, a_2(x) = 0, a_3(x) = x^2, a_4(x) = x$ and $a_5(x) = 1$, we obtain the tetrabonacci polynomials from the row sums of the Riordan array $\mathcal{H}_4(x^3, 0, x^2, x, 1)$. 
Riordan Arrays and Generalized $k$-bonacci numbers

Example

$R_3 = x^9 + 2x^5 + x$

Figure: $(x^3, x^2, 0, x, 1)$-weighted paths and tretranacci polynomials.

Outline

1. Lattice Path - Pascal Matrix - Delannoy Matrix ✓
2. \((a, b, c, d)\)–weighted paths ✓
3. The General Case ✓
4. Schröder Matrix
5. Some additional comments
Delannoy Matrix \rightarrow \text{Inverse Matrix}

\downarrow

Combinatorial Interpretation - Lattice Path
Schröder matrix as inverse of Delannoy matrix

$$\mathcal{H}_2 = \begin{bmatrix}
1 & & & & & \\
1 & 1 & & & & \\
1 & 3 & 1 & & & \\
1 & 5 & 5 & 1 & & \\
1 & 7 & 13 & 7 & 1 & \\
1 & 9 & 25 & 25 & 9 & 1 \\
\vdots & & & & & \\
\end{bmatrix}$$

$$\mathcal{H}_2^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 \\
2 & -3 & 1 & 0 & 0 & 0 \\
-6 & 10 & -5 & 1 & 0 & 0 \\
22 & -38 & 22 & -7 & 1 & 0 \\
-90 & 158 & -98 & 38 & -9 & 1 \\
\vdots & & & & & \\
\end{bmatrix}$$

Delannoy matrix.  Inverse of Delannoy matrix.
Schröder matrix as inverse of Delannoy matrix

\[ \mathcal{H}_2 = \begin{bmatrix}
1 \\
1 & 1 \\
1 & 3 & 1 \\
1 & 5 & 5 & 1 \\
1 & 7 & 14 & 7 & 1 \\
1 & 9 & 25 & 25 & 9 & 1 \\
\vdots & \end{bmatrix} \quad \mathcal{R} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
2 & 3 & 1 & 0 & 0 & 0 \\
6 & 10 & 5 & 1 & 0 & 0 \\
22 & 38 & 22 & 7 & 1 & 0 \\
90 & 158 & 98 & 38 & 9 & 1 \\
\vdots & \end{bmatrix} \]

Delannoy matrix.  Schröder Matrix.

Combinatorial Interpretation?
Schröder matrix as inverse of Delannoy matrix

\[
\mathcal{H}_2 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
b & 1 & 0 & 0 & 0 \\
b^2 & 2b + c & 1 & 0 & 0 \\
b^3 & 3b^2 + 2cb & 3b + 2c & 1 & 0 \\
b^4 & 4b^3 + 3cb^2 & 6b^2 + 6cb + c^2 & 4b + 3c & 1 \\
\vdots
\end{bmatrix}
\]

Generalized Delannoy matrix.

\[
\mathcal{R} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
b & 1 & 0 & 0 & 0 \\
b^2 + cb & 2b + c & 1 & 0 & 0 \\
b^3 + 3cb^2 + 2c^2b & 3b^2 + 5cb + 2c^2 & 3b + 2c & 1 & 0 \\
\vdots
\end{bmatrix}
\]

Generalized Schröder Matrix.

Combinatorial Interpretation?
Schröder matrix

\( \mathcal{J}_3(n, k) := \) denote the set of weighted paths from the point \((0, 0)\) to the point \((n - k, n)\), whose step set is \( R_3 = \{V = (0, 1), D = (1, 1), H_1 = (1, 0)\} \), such that each step is labelled with weights \(c, d, e_1\), respectively; and that its last step is not a horizontal step and that never falling below line \(y = x\).

\( \mathcal{L}_3(n, k) := \) total sum of the weights of all weight paths in \( \mathcal{J}_3(n, k) \).

Schröder matrix
Schröder matrix

**Generalized Schröder Matrix** $S_3(c, d, e_1) = [L_3(n, k)]_{n,k \in \mathbb{N}}$,

The unsigned matrix $H_3^{-1}(1, 1, 1)$ is equal to $S_3(1, 1, 1)$. 
Problem

\[ \mathcal{H}_3(a, b, c, d) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
b & 1 & 0 & 0 \\
b^2 & 2b + c & 1 & 0 \\
b^3 & 3b^2 + 2cb + d & 3b + 2c & 1 \\
b^4 & 4b^3 + 3cb^2 + 2db & 6b^2 + 6cb + c^2 + 2d & 4b + 3c & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots 
\end{bmatrix} \]

Generalized Lattice Matrix (4 steps).

\[ S_3 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
b & 1 & 0 & 0 & 0 \\
b^2 + cb & 2b + c & 1 & 0 & 0 \\
b^3 + 3cb^2 + 2c^2b - db & 3b^2 + 5cb + 2c^2 - d & 3b + 2c & 1 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots 
\end{bmatrix} \]

Generalized Inverse Lattice Matrix (4 Steps).

Combinatorial Interpretation?
Generalized Schröder matrix

\[ \mathcal{J}_4(n, k) := \text{the family of weighted lattice paths from the point } (0, 0) \text{ to the point } (n - k, n), \text{ whose step set is} \]

\[ R_4 = \{ V = (0, 1), D = (1, 1), H_1 = (1, 0), H_2 = (2, 0) \}, \]

such that each step is labelled with weights \( c, d, e_1, e_2 \), respectively, the path never falls below the line \( y = x \), and its last step is not a horizontal step.

\[ \mathcal{L}_4(n, k) := \text{total sum of the weights of all weight paths in } \mathcal{J}_4(n, k). \]

\[ S_4(c, d, e_1, e_2) := (\mathcal{L}_4(n, k))_{n, k \in \mathbb{N}}. \]
Schröder matrix

Example

\[(0, 0) \rightarrow (2, 3)\]
Schröder matrix

\[ \mathcal{L}_4(3, 1) = 2e_1^2 c^3 + e_2 c^3 + 5de_1 c^2 + 3d^2 c. \]
Schröder matrix

\[ \mathcal{L}_4(n+1, k+1) = a_0 \mathcal{L}_4(n, k) + a_1 \mathcal{L}_4(n, k+1) + a_2 \mathcal{L}_4(n+1, k+2) + \cdots + a_n \mathcal{L}_4(n, n), \quad n, k \geq 0; \]

where \( a_0 = c, \quad a_1 = ce_1 + d, \quad a_2 = e_1(ce_1 + d) + e_2 c, \ldots \).
Schröder matrix

In general,

\[ a_n = e_1 a_{n-1} + e_2 a_{n-2} \]

for every integer \( n \geq 2 \).

\[ A^{(4)}(x) = \sum_{i=0}^{\infty} a_i x^i = \frac{c + dx}{1 - e_1 x - e_2 x^2}. \]

If the point is on line \( y = x \), we have

\[ \mathcal{L}_4(n + 1, 0) = b_0 \mathcal{L}_4(n, 0) + b_1 \mathcal{L}_4(n, 1) + b_2 \mathcal{L}_4(n + 1, 2) + \cdots + b_n \mathcal{L}_4(n, n), \quad n, k \geq 0. \]

Note that \( b_0 = d, \quad b_1 = de_1, \quad b_2 = d(e_1^2 + e_2). \ldots \) In general, \( b_n = e_1 b_{n-1} + e_2 b_{n-2} \) for every integer \( n \geq 2 \). Therefore,

\[ Z^{(4)}(x) = \sum_{i=0}^{\infty} b_i x^i = \frac{d}{1 - e_1 x - e_2 x^2}. \]
Schröder matrix

In general,

\[ a_n = e_1 a_{n-1} + e_2 a_{n-2} \]

for every integer \( n \geq 2 \).

\[
A^{(4)}(x) = \sum_{i=0}^{\infty} a_i x^i = \frac{c + dx}{1 - e_1 x - e_2 x^2}.
\]

If the point is on line \( y = x \), we have

\[
L_4(n+1,0) = b_0 L_4(n,0) + b_1 L_4(n,1) + b_2 L_4(n+1,2) + \cdots + b_n L_4(n,n), \quad n, k \geq 0.
\]

Note that \( b_0 = d \), \( b_1 = de_1 \), \( b_2 = d(e_1^2 + e_2) \). In general, \( b_n = e_1 b_{n-1} + e_2 b_{n-2} \) for every integer \( n \geq 2 \). Therefore,

\[
Z^{(4)}(x) = \sum_{i=0}^{\infty} b_i x^i = \frac{d}{1 - e_1 x - e_2 x^2}.
\]
Schröder matrix

In general,

\[ a_n = e_1 a_{n-1} + e_2 a_{n-2} \]

for every integer \( n \geq 2 \).

\[ A^{(4)}(x) = \sum_{i=0}^{\infty} a_i x^i = \frac{c + dx}{1 - e_1 x - e_2 x^2}. \]

If the point is on line \( y = x \), we have

\[ \mathcal{L}_4(n+1, 0) = b_0 \mathcal{L}_4(n, 0) + b_1 \mathcal{L}_4(n, 1) + b_2 \mathcal{L}_4(n + 1, 2) + \cdots + b_n \mathcal{L}_4(n, n), \quad n, k \geq 0. \]

Note that \( b_0 = d, \quad b_1 = d e_1, \quad b_2 = d (e_1^2 + e_2), \ldots \). In general, \( b_n = e_1 b_{n-1} + e_2 b_{n-2} \) for every integer \( n \geq 2 \). Therefore,

\[ Z^{(4)}(x) = \sum_{i=0}^{\infty} b_i x^i = \frac{d}{1 - e_1 x - e_2 x^2}. \]
Schröder matrix

A-sequence and Z-sequence

**Theorem**

If \( c \neq 0 \) then the matrix \( S_4(c, d, e_1, e_2) \) is a Riordan array given by

\[
S_4(c, d, e_1, e_2) = \left( \frac{c}{c + dx}, x \frac{1 - e_1 x - e_2 x^2}{c + dx} \right)^{-1}.
\]

**Theorem**

The inverse of the Riordan matrix \( H_4 \) and the Riordan matrix \( S_4 \) are related by

\[
H_4(1, b, c_1, c_2)^{-1} = S_4(1, -b, -c_1, -c_2).
\]
General Case

\[ J_{m+2}(n, k) := \text{the set of weighted paths from the point } (0, 0) \text{ to the point } (n - k, n), \text{ with step set} \]

\[ R_{m+2} = \{V = (0, 1), D = (1, 1), H_1 = (1, 0), H_2 = (2, 0), \ldots, H_m = (m, 0)\}, \]

where each step is labelled with weights \( c, d, e_1, \ldots, e_m \), respectively, and the path never falls below \( y = x \) and its last step is not horizontal.

\[ L_{m+2}(n, k) := \text{total sum of the weights of all weight paths in} \ J_{m+2}(n, k). \]
Schröder matrix

Lemma

The numbers $L_{m+2}(n, k)$ satisfy the following recurrence relation

$$L_{m+2}(n + 1, k + 1) = a_0 L_{m+2}(n, k) + a_1 L_{m+2}(n, k + 1) + \cdots + a_n L_{m+2}(n, n), \quad n, k \geq 0.$$ 

where $a_n = e_1 a_{n-1} + e_2 a_{n-2} + \cdots + e_m a_{n-m}, \quad n \geq m,$

$$a_0 = c, \quad a_1 = c e_1 + d, \quad a_i = \sum_{j=0}^{i-1} e_{i-j} a_j, \quad 2 \leq i \leq m - 1.$$ 

$L_{m+2}(n + 1, 0) = b_0 L_{m+2}(n, 0) + b_1 L_{m+2}(n, 1) + \cdots + b_n L_{m+2}(n, n),$ 

where $b_n = e_1 b_{n-1} + e_2 b_{n-2} + \cdots + e_m b_{n-m},$ for every $n \geq m$ and

$$b_0 = d, \quad b_i = \sum_{j=0}^{i-1} e_{i-j} b_j, \quad 1 \leq i \leq m - 1.$$
Lemma

The generating functions $A^{(m+2)}(x)$ and $Z^{(m+2)}(x)$ of the sequences $(a_i)_{i \in \mathbb{N}}$ and $(b_i)_{i \in \mathbb{N}}$ are

\[ A^{(m+2)}(x) := \sum_{i=0}^{\infty} a_i x^i = \frac{c + dx}{1 - e_1 x - e_2 x^2 - \cdots - e_m x^m}, \]

and

\[ Z^{(m+2)}(x) := \sum_{i=0}^{\infty} b_i x^i = \frac{d}{1 - e_1 x - e_2 x^2 - \cdots - e_m x^m}. \]
Schröder matrix

Theorem

The infinite matrix $S_{m+2}$ is a Riordan matrix where $A^{(m+2)}(x)$ and $Z^{(m+2)}(x)$ are the generating functions of the A-sequence and Z-sequence, respectively. Moreover, if $c \neq 0$ then the matrix $S_{m+2}(c, d, e_1, e_2, \ldots, e_m)$ is a Riordan array given by

$$S_{m+2}(c, d, e_1, e_2, \ldots, e_m) = \left( \frac{c}{c + dx}, x \frac{1 - e_1 x - e_2 x^2 - \cdots - e_m x^m}{c + dx} \right)^{-1}.$$
Schröder matrix

Theorem

The inverse of the Riordan matrix $\mathcal{H}_{m+2}$ and the Riordan matrix $S_{m+2}$ are related by

$$\mathcal{H}_{m+2}(1,b,c_1,c_2,\ldots,c_m)^{-1} = S_{m+2}(1,-b,-c_1,-c_2,\ldots,-c_m).$$

Outline

1. Lattice Path - Pascal Matrix - Delannoy Matrix ✓
2. \((a, b, c, d)\)–weighted paths ✓
3. The General Case ✓
4. Schröder Matrix ✓
5. Some additional comments
Given a positive real sequence $\mathcal{A} = \{a_n\}_{n \geq 0}$, we say that $\mathcal{A}$ is

- **unimodal** if there exists an integer $0 \leq j$ such that $a_0 \leq \cdots \leq a_{j-1} \leq a_j \geq a_{j+1} \geq \cdots$.
- The sequence $\mathcal{A}$ is called **log-concave** if it satisfies, for all $1 \leq j$, $a_j^2 \geq a_{j-1}a_{j+1}$.
Some additional comments

Example

\[ \{ \binom{n}{k} \}_{0 \leq k \leq n} \]

\[
\begin{align*}
\binom{0}{0} & \\
\binom{1}{0} & \binom{1}{1} \\
\binom{2}{0} & \binom{2}{1} & \binom{2}{2} \\
\binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} \\
\vdots \\
\binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{n}
\end{align*}
\]
Some additional comments

Example
\[ \left\{ \binom{n-k}{k} \right\}_{0 \leq k \leq \lfloor n/2 \rfloor} \]

\[
\begin{array}{llllll}
(0) & (1) & (2) & (3) & \ldots & (n) \\
0 & 0 & 0 & 0 & \ldots & 0 \\
1 & 1 & 1 & 1 & \ldots & 1 \\
2 & 2 & 2 & 2 & \ldots & 2 \\
3 & 3 & 3 & 3 & \ldots & 3 \\
\vdots & & & & & \vdots \\
n & n & n & n & \ldots & n \\
\end{array}
\]

Some additional comments

Example

Belbachir and Szalay (2008): any sequence of binomial coefficients lying along a ray in Pascal’s triangle is log-concave, thus unimodal

\[
\left\{ \binom{u_0 + \alpha k}{v_0 + \beta k} \right\}_k
\]

\[
\begin{array}{cccc}
\binom{0}{0} & \binom{1}{1} & \binom{2}{2} \\
\binom{2}{0} & \binom{2}{1} & \binom{2}{2} \\
\binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} \\
& & \vdots \\
\binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{n} \\
\end{array}
\]

H. Belbachir, L. Szalay. Unimodal rays in the ordinary and generalized Pascal triangles. J. Integer Seq. 11(2008), Article. 08.2.4.


Some additional comments

Example

Stirling Matrix \( \{ \binom{n}{m} \}_{n,k \geq 0} \)
Log-concave: \( \{ \binom{n_0-a_i}{k_0+b_i} \}_{i} \)


Bonus Example!

Example

Adiprasito, Huh and Katz solved the Rota conjecture on the log-concavity of the characteristic polynomial of matroids.
Some additional comments

Example

Stirling Matrix \( \left\{ \binom{n}{m} \right\}_{n,k \geq 0} \)

Log-concave: \( \left\{ \binom{n_0 - a_i}{k_0 + b_i} \right\}_i \)


Bonus Example!

Example

Adiprasito, Huh and Katz solved the *Rota conjecture* on the log-concavity of the characteristic polynomial of matroids.
Some additional comments

Delannoy Matrix?

\[
\mathcal{H}_2 = \begin{bmatrix}
1 & 1 \\
1 & 3 & 1 \\
1 & 5 & 5 & 1 \\
1 & 7 & 13 & 7 & 1 \\
1 & 9 & 25 & 25 & 9 & 1 \\
\vdots & & & & & \\
\end{bmatrix}
\]

Conjecture

Let \( n, p, r \) be positive integers and \( q \in \mathbb{Z} \), with \( n \geq p, 0 \leq p < r \) and \( q + r > 0 \) then the sequence \( \{d(n - qk, p + rk)\}_{0 \leq k} \) satisfies:

- If \( q > 0 \) then the sequence is log-concave and therefore unimodal.
- If \( q < 0 \) and \( q < r \) then the sequence is log-concave and therefore unimodal.
- If \( q < 0 \) and \( q = r \) then the sequence is log-concave and increasing.
- If \( q < 0 \) and \( r < q \) then the sequence is increasing and asymptotically log-convex.
Some additional comments

Theorem

Let \( q \in \mathbb{Z}, \ r \in \mathbb{N}, \ p \in \mathbb{Z}^+ \) with \( 0 \leq p < r \) and \( q + r > 0 \). Any sequence \( \{d(n - qk, p + rk)\}_{0 \leq k \leq \lfloor (n-p)/(q+r) \rfloor} \) lying along all finite rays in the generalized Delannoy matrix is log-concave, thus unimodal.

(a,b,c,d)-Matrix? General case?
Some additional comments

Theorem

Let $q \in \mathbb{Z}$, $r \in \mathbb{N}$, $p \in \mathbb{Z}^+$ with $0 \leq p < r$ and $q + r > 0$. Any sequence $\{d(n - qk, p + rk)\}_{0 \leq k \leq \lfloor (n-p)/(q+r) \rfloor}$ lying along all finite rays in the generalized Delannoy matrix is log-concave, thus unimodal.

(a,b,c,d)-Matrix? General case?
Some additional comments

Delannoy Matrix

\[
\mathcal{H}_2 = \begin{bmatrix}
1 \\
1 & 1 \\
1 & 3 & 1 \\
1 & 5 & 5 & 1 \\
1 & 7 & 13 & 7 & 1 \\
1 & 9 & 25 & 25 & 9 & 1 \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
\end{bmatrix}
\]

- Central Delannoy Numbers:
  \[\{D_n\}_n = \{1, 3, 13, 63, 321, 1683, 8989, 48639, \ldots \}\]
- Liu and Wang, 2007: proved that the central Delannoy sequence is log-convex.
- Given a positive real sequence \(\mathcal{A} = \{a_n\}_{n\geq 0}\), we say that \(\mathcal{A}\) is log-convex if it satisfies, for all \(1 \leq j\), \(a_j^2 \leq a_{j-1}a_{j+1}\).
Some additional comments

Inverse of Delannoy matrix (Schröder matrix)

\[
\mathcal{H}^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & -3 & 1 & 0 & 0 & 0 \\
-6 & 10 & -5 & 1 & 0 & 0 \\
22 & -38 & 22 & -7 & 1 & 0 \\
-90 & 158 & -98 & 38 & -9 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\]

Central sequence:

\[\{F_n\}_n = \{1, 3, 22, 194, 1838, 18082, 182054, 1861890, \ldots \}\]

- \(F_n\) counts the number of lattice paths from the point \((0, 0)\) to the point \((n, 2n)\), whose step set is \(\{V = (0, 1), D = (1, 1), H = (1, 0)\}\) such that its last step is not a horizontal step and that never falling below line \(y = x\).
Some additional comments

Inverse of Delannoy matrix (Schröder matrix)

\[ H^{-1} = \begin{bmatrix}
  1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 \\
 2 & -3 & 1 & 0 & 0 & 0 & 0 \\
-6 & 10 & -5 & 1 & 0 & 0 & 0 \\
 22 & -38 & 22 & -7 & 1 & 0 & 0 \\
-90 & 158 & -98 & 38 & -9 & 1 & 0 \\
  \vdots &  \vdots &  \vdots &  \vdots &  \vdots &  \vdots &  \ddots
\end{bmatrix}. \]

▷ Central sequence:

\[ \{F_n\}_n = \{1, 3, 22, 194, 1838, 18082, 182054, 1861890, \ldots \} \]

▷ \(F_n\) counts the number of lattice paths from the point \((0, 0)\) to the point \((n, 2n)\), whose step set is \(\{V = (0, 1), D = (1, 1), H = (1, 0)\}\) such that its last step is not a horizontal step and that never falling below line \(y = x\).
Some additional comments

\[ F_2 = 22 \]

**Figure:** Combinatorial interpretation for \( F_n \).
Some additional comments

**Theorem**

*The sequence $F_n$ is *log-convex*.*

**Proof.**

Computational Method...

$$F_n = \sum_{j=0}^{n} \left( \frac{n}{2n-j} \binom{2n-1}{j} \binom{3n-j-1}{2n-1} \right) + \frac{n+1}{2n-j} \binom{2n-1}{j} \binom{3n-j-2}{2n-1}, \quad n \geq 0$$
Some additional comments

**Theorem**

The sequence $F_n$ is log-convex.

**Proof.**

Computational Method...

$$F_n = \sum_{j=0}^{n} \left( \frac{n}{2n-j} \binom{2n-1}{j} \binom{3n-j-1}{2n-1} + \frac{n+1}{2n-j} \binom{2n-1}{j} \binom{3n-j-2}{2n-1} \right), \quad n \geq 0.$$
Some additional comments

From the Kauers’s multivariate guessing library we obtain that

\[ p_n F_n = q_n F_{n-1} + r_n F_{n-2}, \quad n \geq 2 \]

with the initial values \( F_0 = 1 \) and \( F_1 = 3 \), and

\[ p_n = 20n^4 - 72n^3 + 85n^2 - 39n + 6, \]
\[ q_n = 220n^4 - 902n^3 + 1272n^2 - 711n + 129, \]
\[ r_n = 20n^4 - 92n^3 + 89n^2 + 36n - 45. \]
Some additional comments

Let \( \{z_n\}_n \) be a sequence such that

\[
a(n)z_{n+1} = b(n)z_n + c(n)z_{n-1},
\]

(1)

where \( a(n), b(n) \) and \( c(n) \) are positive for \( n \geq 1 \).

Lemma (Liu and Wang, 2007)

Let \( \{z_n\}_n \) be defined by (1) and

\[
\lambda_n = \frac{b(n) + \sqrt{b^2(n) + 4a(n)c(n)}}{2a(n)}.
\]

Suppose that \( z_0, z_1, z_2, z_3 \) is log-convex and that the inequality

\[
a(n)\lambda_{n-1}\lambda_{n+1} - b(n)\lambda_{n-1} - c(n) \geq 0
\]

is true for \( n \geq 2 \). Then the sequence \( \{z_n\}_n \) is log-convex.

Wolfram Mathematica
\((1, 1, 1, 1)\)-weighted paths

\[
\mathcal{H}_3(1, 1, 1, 1) = 
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 3 & 1 & 0 & 0 & 0 \\
1 & 6 & 5 & 1 & 0 & 0 \\
1 & 9 & 15 & 7 & 1 & 0 \\
1 & 12 & 33 & 28 & 9 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

Central sequence: \( \left\{ d_{2n,n}^{(3)} \right\}_{n \in \mathbb{N}} = \{1, 3, 15, 81, 459, 2673, \ldots \} \)
(1, 1, 1, 1)-weighted paths

Theorem (Barry, 2013)

Let \((d(t), h(t)) = (d(t), tf(t))\) be an element of the Riordan group of matrices \(\mathcal{R}\) (with \(f(0) \neq 0\)). Let \(d_{n,k}\) denote the \((n, k)\)-th element of this matrix, and let \(v(t)\) denote the power series (with \(v(0) = 0\))

\[
v(t) = \left( \frac{t}{f(t)} \right).
\]

Then the generating function of the central term sequence \(d_{2n,n}\) is given by

\[
\frac{d(v(t))}{f(v(t))} \frac{d}{dt}v(t).
\]

The generating function of the central sequence \(\left\{d_{2n,n}^{(3)}\right\}_{n \in \mathbb{N}} = \{1, 3, 15, 81, 459, 2673, \ldots \}\) is

\[
\sum_{n=0}^{\infty} d_{2n,n}^{(3)} z^n = \sum_{n=0}^{\infty} D_3(n, n) z^n = \frac{1}{\sqrt{1 - 6z - 3z^2}}.
\]
**Theorem (Barry, 2013)**

Let \((d(t), h(t)) = (d(t), tf(t))\) be an element of the Riordan group of matrices \(R\) (with \(f(0) \neq 0\)). Let \(d_{n,k}\) denote the \((n, k)\)-th element of this matrix, and let \(v(t)\) denote the power series (with \(v(0) = 0\))

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\]

The generating function of the central sequence \(\left\{ d^{(3)}_{2n,n} \right\}_{n \in \mathbb{N}} = \{1, 3, 15, 81, 459, 2673, \ldots \}\) is

\[
\sum_{n=0}^{\infty} d^{(3)}_{2n,n} z^n = \sum_{n=0}^{\infty} D_3(n, n) z^n = \frac{1}{\sqrt{1 - 6z - 3z^2}}.
\]
$(1, 1, 1, 1)$-weighted paths

- Grand Motzkin path:
(1, 1, 1, 1)-weighted paths

\( g_n^{(3, 3)} := \) the number of grand Motzkin path such that each horizontal step is colored with one of 3 colors and each up diagonal step is colored with one of 3 specific colors.

**Corollary**

The number of \( 3^H 3^U \)-Grand Motzkin path is equal to the number of \((1, 1, 1, 1)\)-weighted paths from the point \((0, 0)\) to the point \((n, n)\), i.e.,

\[ g_n^{(3, 3)} = D_3(n, n). \]
$$(1, 1, 1, 1)$$-weighted paths

Dziemiańczuk, 2013:
Let $S(n, k)$ be the number of lattice paths from $(0, 0)$ to $(n, k)$ in the plane $\mathbb{Z} \times \mathbb{N}$ with set of steps

$$S = \{ H = (1, 0), V = (0, 1), \\ D_1 = (1, 1), D_2 = (-1, 1) \}$$

Therefore

$$\sum_{n=0}^{\infty} S(0, n) z^n = \frac{1}{\sqrt{1 - 6z - 3z^2}}.$$ 

Therefore

$$S(0, n) = g_n^{(3,3)} = D_3(n, n).$$
(1, 1, 1, 1)-weighted paths
(1, 1, 1, 1)-weighted paths
$(1, 1, 1, 1)$-weighted paths
Some additional comments

- Is there any Bijection?

\[ S = \{ H = (1,0), V = (0,1), \]
\[ D_1 = (1,1), D_2 = (-1,1) \} \]

\[ R(n,k) = \left[z^k\right] \frac{2^{n+2}(1+z)^n\sqrt{1-6z-3z^2}}{(1-z+\sqrt{1-6z-3z^2})^{n+2} - (1-z-\sqrt{1-6z-3z^2})^{n+2}}, \]

where \( R(n,k) \) is the family of lattice paths from the origin to \((n,k)\) whose points lie entirely in the integer rectangle of lattice points \( \{(i,j) : 0 \leq i \leq n, 0 \leq j \leq k \} \).
Some additional comments


\[
A(n, r) = A(n-1, r-1) + q^n A(n, r-1) + q^r A(n-1, r)
\]

Some additional comments

The valleys and peaks of a Dyck path $P$ are the local minima and local maxima, respectively. We say that a Dyck path $P$ is non-decreasing if the $y$-coordinates of valleys of the path $P$ form a non-decreasing sequence.
Some additional comments

Non-decreasing sequence Dyck path.

The number of non-decreasing Dyck paths of length $2n$ is the Fibonacci number $F_{2n-1}$.

Deutsch and Prodingher, 2003, give a bijection between the non-decreasing Dyck path of length $2n$ and the direct column-convex polyomino (dccp) of area $n$. 
The valleys of a Motzkin path $P$ are the local minima. We say that a Motzkin path $P$ is non-decreasing if the $y$-coordinates of all valleys (including also the left valleys and the right valleys) of the path $P$ form a non-decreasing sequence.
Some additional comments

- $l(P) :=$ the length of $P$,
- $h(P) :=$ the number of horizontal steps of $P$
- $r(P) :=$ the number of rises (up-diagonal steps) of $P$
- $p(P) :=$ the number of peaks of $P$.

$$F(x, y, z, q) := \sum_{P \in \mathbb{N}^M} x^{l(P)} y^{h(P)} z^{r(P)} q^{p(P)}.$$  

**Theorem**

The generating function $F(x, y, z, q)$ is given by the following equation

$$F(x, y, z, q) = \frac{x(xy - 1)(xzq + y)(x^2z - 1)}{1 - 2xy + x^2y^2 - 2x^2z - qx^2z + 2x^3yz + qx^3yz - x^4y^2z + x^4z^2 - x^5yz^2}.$$
Some additional comments

- $l(P) :=$ the length of $P$,
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**Theorem**

*The generating function $F(x, y, z, q)$ is given by the following equation*

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F(x, y, z, q) = \frac{x(xy - 1)(xzq + y)(x^2z - 1)}{1 - 2xy + x^2y^2 - 2x^2z - qx^2z + 2x^3yz + qx^3yz - x^4y^2z + x^4z^2 - x^5yz^2}.
\]
Some additional comments

Proof.
Symbolic Method:

Figure: Factorizations of any non-decreasing Motzkin path.
Some additional comments

If $y = 1 = z = q$ we obtain the generating function for the non-decreasing Motzkin paths respect to the length:

$$F(x, 1, 1, 1) = \frac{x(x^2 - 1)^2}{1 - 2x - 2x^2 + 3x^3 - x^5} = x + 2x^2 + 4x^3 + 9x^4 + 21x^5 + 49x^6 + 115x^7 + \cdots$$

If $y = 0$ and $q = 1 = z$ then

$$F(x, 0, 1, 1) = \frac{x^2(1 - x^2)}{1 - 3x^3 + x^4} = \sum_{n=1}^{\infty} F_{2n-1}x^{2n}.$$
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Some additional comments

- A path which is a prefix of a non-decreasing Motzkin path is called **non-decreasing Motzkin prefix**.
- The prefixes of the classical Dyck paths are called **Ballot paths**.
- \( m_{n,k} \) = number of non-decreasing Motzkin prefixes of length \( n \) and height \( k \).
- \( \bar{M} = [m_{n,k}]_{n,k \geq 0} \):

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\
4 & 5 & 3 & 1 & 0 & 0 & 0 & 0 \\
9 & 12 & 9 & 4 & 1 & 0 & 0 & 0 \\
21 & 30 & 25 & 14 & 5 & 1 & 0 & 0 \\
49 & 74 & 69 & 44 & 20 & 6 & 1 & 0 \\
115 & 182 & 185 & 133 & 70 & 27 & 7 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
\]
Some additional comments

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\[
M = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\
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9 & 12 & 9 & 4 & 1 & 0 & 0 & 0 \\
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49 & 74 & 69 & 44 & 20 & 6 & 1 & 0 \\
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\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
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$$
\mathbf{M} = 
\begin{pmatrix}
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1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
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9 & 12 & 9 & 4 & 1 & 0 & 0 & 0 \\
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- $\mathcal{M} = [m_{n,k}]_{n,k \geq 0}$.

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1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\
4 & 5 & 3 & 1 & 0 & 0 & 0 & 0 \\
9 & 12 & 9 & 4 & 1 & 0 & 0 & 0 \\
21 & 30 & 25 & 14 & 5 & 1 & 0 & 0 \\
49 & 74 & 69 & 44 & 20 & 6 & 1 & 0 \\
115 & 182 & 185 & 133 & 70 & 27 & 7 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
\]
Some additional comments

- A path which is a prefix of a non-decreasing Motzkin path is called non-decreasing Motzkin prefix.
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- \( M = [m_{n,k}]_{n,k \geq 0} \).

\[
M = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\
4 & 5 & 3 & 1 & 0 & 0 & 0 & 0 \\
9 & 12 & 9 & 4 & 1 & 0 & 0 & 0 \\
21 & 30 & 25 & 14 & 5 & 1 & 0 & 0 \\
49 & 74 & 69 & 44 & 20 & 6 & 1 & 0 \\
115 & 182 & 185 & 133 & 70 & 27 & 7 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
\]
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- $\mathbb{M} = [m_{n,k}]_{n,k \geq 0}$.

$$
\mathbb{M} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\
4 & 5 & 3 & 1 & 0 & 0 & 0 & 0 \\
9 & 12 & 9 & 4 & 1 & 0 & 0 & 0 \\
21 & 30 & 25 & 14 & 5 & 1 & 0 & 0 \\
49 & 74 & 69 & 44 & 20 & 6 & 1 & 0 \\
115 & 182 & 185 & 133 & 70 & 27 & 7 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots 
\end{pmatrix}
$$
Some additional comments

Example

$m_{4,2} = 9$ paths.

Figure: Non-decreasing Motzkin prefixes of length 4 and height 2.
Some additional comments

Theorem

Let \( T_k(x) \) be the generating function defined by

\[
T_k(x) := \sum_{n=0}^{\infty} d_{n,k} x^n.
\]

Then

\[
T_k(x) = \frac{1 - x - 2x^2 + x^3}{1 - 2x - 2x^2 + 3x^3 - x^5} \left( x \frac{(1 - x)^2(1 + x)}{1 - 2x - x^2 + 2x^3 - x^4} \right)^k.
\]
Theorem

The matrix $M$ is a Riordan array given by

$$M = \begin{pmatrix}
\frac{1 - x - 2x^2 + x^3}{1 - 2x - 2x^2 + 3x^3 - x^5}, & x \frac{(1 - x)^2(1 + x)}{1 - 2x - x^2 + 2x^3 - x^4}
\end{pmatrix}.$$
Some additional comments

Theorem

*The matrix $\mathbf{M}$ is a Riordan array given by*

$$
\mathbf{M} = \left( \frac{1 - x - 2x^2 + x^3}{1 - 2x - 2x^2 + 3x^3 - x^5}, x \frac{(1 - x)^2(1 + x)}{1 - 2x - x^2 + 2x^3 - x^4} \right).
$$

Additional Properties of the non-decreasing Motzkin paths....
Thank you!!!
Gracias!!

Figure: Churros and Chocolate; Chocolatería San Ginés, 17-Jul-2017!