Stieltjes and Hamburger moment sequences in combinatorics

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Background

The Problem of Moments

continued fractions

Thomas Joannes Stieltjes (1856-1894)

- Widder, The Laplace Transform, 1941.
- Shohat and Tamarkin, The Problem of Moments, 1943.
Stieltjes and Hamburger moment sequence

**Definition**

Let $\alpha = (a_n)_{n \geq 0}$ be an infinite sequence of real numbers. Call $\alpha$ a Stieltjes (Hamburger, resp.) moment (SM (HM) for short) sequence if it has the form

$$a_n = \int_0^{+\infty} x^n \, d\mu(x),$$

where $\mu$ is a positive Borel measure on $x \geq 0$ ($x \in \mathbb{R}$, resp.).

**Example**

The factorial numbers $n!$ form a SM,

$$n! = \int_0^{\infty} x^n \, d(1 - e^{-x}).$$
Let $\alpha = (a_n)_{n \geq 0}$ be real numbers. Define the **Hankel matrix** $H(\alpha)$ of $\alpha$ by

$$H(\alpha) := [a_{i+j}]_{i,j \geq 0} = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$  

Call a matrix **totally positive** if all its minors are nonnegative.

**Theorem**

$\alpha$ is SM $\iff$ $H(\alpha)$ and $H(\overline{\alpha})$ are positive semi-definite $\iff$ $H(\alpha)$ is totally positive.

**Example**

Catalan numbers $C_n = \frac{1}{n+1} \binom{2n}{n}$ form a SM.

$$\det[C_{i+j}]_{0 \leq i,j \leq n} = \det[C_{i+j+1}]_{0 \leq i,j \leq n} = 1.$$
**Classical results of Hamburger moment sequence**

**Theorem**

Let \( \alpha = (a_n)_{n \geq 0} \) be an infinite sequence of real numbers.

\[ \alpha \text{ is HM} \iff H(\alpha) \text{ is positive semi-definite} \iff \sum_{i,j \geq 0} a_{i+j}x_i x_j \geq 0. \]

- \( \alpha = (a_n)_{n \geq 0} \) is SM

\[ \iff \text{Both } \alpha = (a_n)_{n \geq 0} \text{ and } \bar{\alpha} = (a_{n+1})_{n \geq 0} \text{ are HM}. \]

- SM \( \Rightarrow \) HM.

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### Classical results of Hamburger moment sequence

**Theorem**

Let \( \alpha = (a_n)_{n \geq 0} \) be an infinite sequence of real numbers. Then \( \alpha \) is a HM sequence if all Hankel determinants

\[
h_n(\alpha) := \det [a_{i+j}]_{0 \leq i, j \leq n}
\]

are nonnegative and \( h_N = 0 \) implies \( h_n = 0 \) for \( n \geq N \).

**Example**

The Fibonacci sequence \( \mathcal{F} = (F_n)_{n \geq 0} \) is HM.

\[
F_0 = F_1 = 1 \text{ and } F_n = F_{n-1} + F_{n-2},
\]

Catalan-like numbers

Definition

Let \( s = (s_k)_{k \geq 0} \) and \( t = (t_{k+1})_{k \geq 0} \) be nonnegative sequences. Define the recursive matrix \( A^{(s,t)} = [a_{n,k}]_{n,k \geq 0} \) by

\[
a_{0,0} = 1, \quad a_{n+1,k} = a_{n,k-1} + s_k a_{n,k} + t_{k+1} a_{n,k+1},
\]

where \( a_{n,k} = 0 \) unless \( n \geq k \geq 0 \).

Call \( a_n := a_{n,0} \) the Catalan-like numbers.
Example of recursive matrix

\[ A^{(s,t)} = \begin{bmatrix}
1 & 1 \\
1 & 3 & 1 \\
2 & 5 & 9 & 5 & 1 \\
5 & 28 & 20 & 7 & 1 \\
14 & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{bmatrix}, \]

where \( s = (1, 2, 2, \ldots) \), \( t = (1, 1, 1, \ldots) \).

- \( C_n = (1, 1, 2, 5, 14, \ldots) \).
Example of recursive matrix

$$A^{(s,t)} = \begin{bmatrix}
1 & 1 \\
2 & 2 & 1 \\
4 & 5 & 3 & 1 \\
9 & 12 & 9 & 4 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix},$$

where $s = t = (1, 1, 1, \ldots)$.

- $M_n = (1, 1, 2, 4, 9, \ldots)$. 
Rewrite the recursive relation (1) as

\[
\begin{bmatrix}
a_{1,0} & a_{1,1} \\
a_{2,0} & a_{2,1} & a_{2,2} \\
a_{3,0} & a_{3,1} & a_{3,2} & a_{3,3} \\
\vdots & \ddots & \ddots & \ddots
\end{bmatrix}
= \begin{bmatrix}
a_{0,0} & a_{1,1} \\
a_{1,0} & a_{1,1} \\
a_{2,0} & a_{2,1} & a_{2,2} \\
\vdots & \ddots & \ddots & \ddots
\end{bmatrix} \begin{bmatrix}
s_0 & 1 \\
t_1 & s_1 & 1 \\
t_2 & s_2 & \ddots \\
\vdots & \ddots & \ddots
\end{bmatrix},
\]

or briefly,

\[
\overline{A} = AJ
\]

where \( \overline{A} \) is obtained from \( A \) by deleting the 0th row and \( J \) is the tridiagonal matrix

\[
J := J^{\sigma,\tau} = \begin{bmatrix}
s_0 & 1 \\
t_1 & s_1 & 1 \\
t_2 & s_2 & \ddots \\
\vdots & \ddots & \ddots
\end{bmatrix}.
\]
Examples of Catalan-like numbers

- Bell numbers $B_n$: $s = t = (1, 2, 3, ...)$.
- Motzkin numbers $M_n$: $s = t = (1, 1, 1, ...)$.
- Fine numbers $F_n$: $s = (0, 2, 2, ...)$, $t = (1, 1, 1, ...)$.
- Catalan numbers $C_n$: $s = (1, 2, 2, ...)$, $t = (1, 1, 1, ...)$.
- Riordan numbers $R_n$: $s = (0, 1, 1, ...)$, $t = (1, 1, 1, ...)$.
- Factorial numbers $n!$: $s = (1, 3, 5, ...)$, $t = (1, 4, 9, ...)$.
- Large Schröder numbers $r_n$: $s = (2, 3, 3, ...)$, $t = (2, 2, 2, ...)$.
- Little Schröder numbers $S_n$: $s = (1, 3, 3, ...)$, $t = (2, 2, 2, ...)$.
- Central Delannoy numbers $D_n$: $s = (3, 3, 3, ...)$, $t = (4, 2, 2, ...)$.
- Central trinomial coefficients $T_n$: $s = (1, 1, 1, ...)$, $t = (2, 1, 1, ...)$.
- Central binomial coefficients $\binom{2n}{n}$: $s = (2, 2, 2, ...)$, $t = (2, 1, 1, ...)$.
Theorem

The Catalan-like numbers form a HM sequence if and only if all \( t_k \) are nonnegative.

Proof. By the definition and lattice path proofs, we have

\[
a_{m+n} = \sum_{k \geq 0} a_{m,k} a_{n,k} T_k \iff H = ATA^t
\]

where \( T = \text{diag}(T_0, T_1, \ldots) \) and \( T_0 = 1, T_k = t_1 t_2 \cdots t_k \) for \( k \geq 1 \). Hankel determinants of the Catalan-like numbers \( a_n \):

\[
\det[a_{i+j}]_{0 \leq i,j \leq n} = T_0 T_1 \cdots T_n = t_1^n t_2^{n-1} \cdots t_{n-1}^2 t_n.
\]
For $\alpha = (a_n)_{n \geq 0}$, define $L(\alpha) = (b_n)_{n \geq 0}$ by

$$b_n = a_n a_{n+2} - a_{n+1}^2.$$

Then $\alpha$ is log-convex $\iff$ $L(\alpha)$ is nonnegative.


Call $\alpha$ **infinitely log-convex** if $L^m(\alpha)$ is nonnegative for all $m \geq 1$.

**Conjecture (Chen and Xia, PAMS, 2011)**

*The sequence of the large Schröder numbers is $\infty$-log-convex.*


*The Stieltjes moment sequence is $\infty$-log-convex.*
$\infty$-convex and Hamburger moment sequences

For $\alpha = (a_n)_{n \geq 0}$, define $\mathcal{L}(\alpha) = (c_n)_{n \geq 0}$ by

$$c_n = a_n + a_{n+2} - 2a_{n+1}.$$ 

Then $\alpha$ is convex $\iff \mathcal{L}(\alpha)$ is nonnegative.

Definition

Call $\alpha$ $\infty$-convex if $\mathcal{L}(\alpha)$ is nonnegative for all $m \geq 1$.

Theorem

*The Hamburger moment sequence is $\infty$-convex.*
The Apéry numbers

\[ A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \quad n = 0, 1, 2, \ldots, \]

were introduced by Apéry in his famous proof to the irrationality of

\[ \zeta(3) = \sum_{n \geq 1} n^{-3}. \]

**Conjecture (Chen and Xia, PAMS, 2011)**

\((A_n)_{n \geq 0}\) is \(\infty\)-log-convex.

**Conjecture (Sokal, MIT, 2014)**

\((A_n)_{n \geq 0}\) is SM.
Thank you for your attention!